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Abstract

We consider a one–dimensional Bose-Einstein condensate in a infinite square-well (box) potential. This is a nonlinear control system in which the state is the wave function of the Bose Einstein condensate and the control is the length of the box. We prove that local exact controllability around the ground state (associated with a fixed length of the box) holds generically with respect to the chemical potential μ; i.e. up to an at most countable set of μ–values. The proof relies on the linearization principle and the inverse mapping theorem, as well as ideas from analytic perturbation theory.

Key words: quantum systems, controllability of PDEs.

Subject classifications: 35Q55, 35Q93

1 Introduction

1.1 Background and original problem

Controlled manipulation of Bose Einstein condensates (BECs) is an important objective in quantum control theory. In this paper we consider a one-dimensional condensate in a hard-wall trap ("condensate-in-a-box"), where the trap size (box length) is a time-dependent function $L(\tau)$, which provides the control. The model (see (1) below) was first proposed by Band, Malomed, and Trippenbach [4] to study adiabaticity in a nonlinear quantum system. More recently, the opposite regime, fast transitions ("shortcuts to adiabaticity"), has been investigated for BECs in box potentials [44, 24]. Condensates in a box trap have also been realized experimentally [37], an achievement that attracted considerable attention. Motivated by these developments, we study the controllability of the following system [4]

$$\begin{cases} i\hbar\partial_\tau\Phi(\tau,z) = -\frac{\hbar^2}{2m}\partial_z^2\Phi(\tau,z) \mp \kappa|\Phi|^2\Phi(\tau,z), & z\in(0,L(\tau)), \tau\in(0,\tau^*), \\ \Phi(\tau,0)=\Phi(\tau,L(\tau))=0, & \tau\in(0,\tau^*). \end{cases} \tag{1}$$

Here $\hbar$ is Planck's constant, m is the particle mass, $\kappa>0$ is a nonlinearity parameter derived from the scattering length and the particle number, $\tau^*>0$ is a positive real number and $L\in C^0([0,\tau^*],\mathbb{R}_+^*)$ is the length of the box. The '$-$' sign in the right-hand side refers to the focusing case (attractive two-particle interaction), while the '$+$' sign refers to the defocusing

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one (repulsive interaction). In this article, we will work with classical solutions (point-wise solutions) of the system (1).

System (1) is a *nonlinear control system* in which

(i) the *state* is the wave function $\Phi(\tau, z)$, which is normalized

$$\int_0^{L(\tau)} |\Phi(\tau, z)|^2 dz = 1, \quad \forall \tau \in (0, \tau^*); \tag{2}$$

(ii) the *control* is the length L of the box, with

$$L(0) = L(\tau^*) = 1. \tag{3}$$

This problem is a nonlinear variant of the control problem studied by K. Beauchard in [9][1].

1.2 Change of variables

Following Band et al. [4] we introduce new variables,

$$t := \frac{\hbar}{2m} \int_0^\tau \frac{ds}{L(s)^2}, \qquad x := \frac{z}{L(\tau)}, \qquad \Phi(\tau, z) = \frac{\hbar}{\sqrt{2\kappa m} L(\tau)} \psi(t, x), \tag{4}$$

to non-dimensionalize the problem and to transform it to the time-independent domain $(0, 1)$. Then defining

$$u(t) := \frac{2m}{\hbar} \dot{L}(\tau) L(\tau) \tag{5}$$

or, equivalently

$$L(\tau) = \exp\left(\int_0^t u(s) ds \right), \tag{6}$$

we obtain

$$\begin{cases} i\partial_t \psi = -\partial_x^2 \psi \mp |\psi|^2 \psi + iu(t)\partial_x[x\psi], & x \in (0,1), t \in (0,T), \\ \psi(t,0) = \psi(t,1) = 0, & t \in (0,T), \end{cases} \tag{7}$$

where

$$T := \int_0^{\tau^*} \frac{ds}{L(s)^2}. \tag{8}$$

The system (7) is a control system in which

(i) the state is ψ with

$$\|\psi(t)\|_{L^2(0,1)} = \|\psi(0)\|_{L^2(0,1)} e^{\frac{1}{2}\int_0^t u},$$

(ii) the control is the real valued function u.

Note that the previous changes of variables impose constraints on the control u. Indeed, the requirement $L(0) = L(\tau^*) = 1$, together with (6) and (8) impose

$$\int_0^T u = 0.$$

In this article, we will work with classical solutions of (7), that will provide classical solutions of (1).

To ensure that the controllability of (7) gives the one of (1), we need the surjectivity of the map $L \mapsto u$, which is proved in the next proposition.

[1]The study of the controllability of *nonlinear* Schrödinger equations was proposed by Zuazua [49].

Proposition 1 *Let $T>0$, $u \in L^\infty(0,T;\mathbb{R})$ extended by zero on $(-\infty,0)\cup(T,\infty)$ and such that $\int_0^T u(t)dt=0$. The unique maximal solution of the Cauchy problem*

$$\begin{cases} g'(\tau)=\frac{\hbar}{2m}e^{-2\int_0^{g(\tau)}u(s)ds}, \\ g(0)=0, \end{cases} \tag{9}$$

is defined for every $\tau \geqslant 0$, strictly increasing and satisfies

$$\lim_{\tau\to+\infty} g(\tau)=+\infty. \tag{10}$$

thus $\tau^* := g^{-1}(T)$ is well defined. Let $L:[0,\infty)\to[0,\infty)$ be defined by

$$L(\tau):=\exp\left(\int_0^{g(\tau)} u(s)ds\right). \tag{11}$$

Then, (3) and (5) are satisfied.

Proof of Proposition 1: The function $F:\mathbb{R}\to\mathbb{R}$ defined by $F(x):=\frac{\hbar}{2m}e^{-2\int_0^x u(s)ds}$ is continuous, globally Lipschitz (because $u\in L^\infty$), and uniformly bounded. By Cauchy-Lipschitz (or Picard Lindelof) theorem, there exists a unique solution to (9), defined for every $\tau\in[0,+\infty)$. It is strictly increasing on $[0,+\infty)$ because $g'>0$. Now, we prove (10) by contradiction. We assume that $g(\tau)\leqslant T$ for every $\tau\in[0,+\infty)$. Then,

$$g'(\tau)\geqslant \frac{\hbar}{2m}e^{-2\|u\|_\infty T}, \forall \tau\in(0,+\infty)$$

thus

$$T\geqslant g(\tau)\geqslant \frac{\tau\hbar}{2m}e^{-2\|u\|_\infty T}, \forall\tau\in(0,+\infty)$$

which is impossible. Therefore, there exists $\tau_1>0$ such that $g(\tau_1)>T$. Then, $g'\equiv\hbar/2m$ on (τ_1,∞), which implies (10). The relation (3) is satisfied because $g(\tau^*)=T$ and $\int_0^T u=0$. By integrating the first equality of (9) and using (11), we get

$$g(\tau)=\frac{\hbar}{2m}\int_0^\tau \exp\left(-2\int_0^{g(s)}u\right)ds=\frac{\hbar}{2m}\int_0^\tau \frac{ds}{L(s)^2}.$$

Thanks to (11) and (9), we have

$$\frac{2m}{\hbar}\dot{L}(\tau)L(\tau)=\frac{2m}{\hbar}g'(\tau)u[g(\tau)]\exp\left(2\int_0^{g(\tau)}u\right)=u[g(\tau)]$$

which proves (5). $\square$

1.3 Main result

We introduce the unitary $L^2((0,1),\mathbb{C})$ sphere $\mathcal{S}$, the operator A defined by

$$D(A):=H^2\cap H^1_0((0,1),\mathbb{C}), \quad A\varphi:=-\varphi'',$$

and the spaces

$$H^s_{(0)}((0,1),\mathbb{C}):=D(A^{s/2}), \forall s>0. \tag{12}$$

In particular,

$$H^3_{(0)}((0,1),\mathbb{C})=\{\varphi\in H^3((0,1),\mathbb{C});\varphi=\varphi''=0 \text{ at } x=0,1\}.$$

We also introduce, for $T > 0$, the space

$$\dot{H}_0^1((0,T),\mathbb{R}) := \left\{ u \in H_0^1((0,T),\mathbb{R}); \int_0^T u(t)dt = 0 \right\}.$$

For $\mu \in (\mp\pi^2, +\infty)$, we denote by ϕ_μ the nonlinear ground state; i.e. the unique positive solution of the boundary value problem

$$\begin{cases} \phi_\mu'' \pm \phi_\mu^3 = \pm\mu\phi_\mu, & x \in (0,1), \\ \phi_\mu(0) = \phi_\mu(1) = 0. \end{cases} \tag{13}$$

(See Section 2 for existence and properties of ϕ_μ). Then the couple $(\psi_\mu(t,x) := \phi_\mu(x)e^{\pm i\mu t}, u \equiv 0)$ is a trajectory of (7). The goal of this article is to prove the local exact controllability of system (7) around this reference trajectory, for generic $\mu \in (\mp\pi^2, +\infty)$.

Theorem 2 *Let $T > 0$. There exists a countable set $J \subset (\mp\pi^2, +\infty)$ such that, for every $\mu \in (\mp\pi^2, +\infty) \setminus J$, the system (7) is exactly controllable in time T, locally around the ground state; i.e., there exists $\delta = \delta(\mu, T) > 0$ and a C^1–map*

$$\Upsilon : \mathcal{V} \to \dot{H}_0^1((0,T),\mathbb{R}),$$

where

$$\mathcal{V} := \{\psi_f \in H^3_{(0)}((0,1),\mathbb{C}); \|\psi_f - \phi_\mu e^{\pm i\mu T}\|_{H^3_{(0)}} < \delta \text{ and } \|\psi_f\|_{L^2} = \|\phi_\mu\|_{L^2}\},$$

such that, $\Upsilon(\phi_\mu e^{\pm i\mu T}) = 0$, and for every $\psi_f \in \mathcal{V}$, the solution of (7) associated with the control $u := \Upsilon(\psi_f)$, and the initial condition

$$\psi(0,x) = \phi_\mu(x), \quad x \in (0,1) \tag{14}$$

is defined on $[0,T]$ and satisfies $\psi(T) = \psi_f$.

Remark 3 *Note that by the time reversibility of the Schrödinger equation this result may be generalized to include arbitrary initial data $\psi(0,.) = \psi_0$, which are close enough to ϕ_μ in $H^3_{(0)}((0,1),\mathbb{C})$.*

1.4 Structure of this article

The proof of Theorem 2 relies on a linearization principle, which involves proving the controllability of the linear system that arises by linearizing (7) around the trajectory $(\psi_\mu(t,x) := \phi_\mu(x)e^{\pm i\mu t}, u \equiv 0)$ and applying the inverse mapping theorem. Accordingly, this article is organized as follows.

After stating the existence and uniqueness of the ground state (Section 2), i.e. the positive solution ϕ_μ of (13), we study in Section 3 the well–posedness of the Cauchy problem associated with (7). The C^1–regularity of the end–point map is established in Section 4. Section 5 contains a detailed description of the spectral properties of the linearized system. In Section 6, we prove the controllability of the linearized system, under appropriate assumptions ***(A)*** and ***(B)***, which, in Section 7, are shown to hold generically with respect to the chemical potential $\mu \in (\mp\pi^2, +\infty)$. Finally, in Section 8, we prove the main result of this article. The final section of the main part of the paper contains some concluding remarks and perspectives (Section 9).

The main body of the article is followed by four appendices, containing proofs omitted in the main part of the paper to improve its readability. In Appendix A the proof of Proposition 4 (Section 2) on the existence of ground states is provided. The spectral properties of the linearization stated in Section 5 are established in Appendix B. Appendix C contains the proof of the analyticity of the spectrum. Finally, Appendix D deals with trigonometric moment problems: a classical result, which is used in Section 6, is recalled here.

1.5 A brief review of infinite-dimensional bilinear control systems

In this section we provide references to some of the pertinent literature. We do not, however, attempt a comprehensive review of the field, which is beyond the scope of the this paper[2].

Early controllability results for Schrödinger equations with bilinear controls were negative; see [30, 39, 43] and in particular [45] obtained by Turinici as a corollary to a more general result by Ball, Marsden and Slemrod [3]. Turinici's result was adapted to nonlinear Schrödinger equations by Illner, Lange and Teismann [31]. Because of these non–controllability properties, bilinear Schrödinger equations were considered to be non–controllable for a long time. However, some progress was eventually made and the question is now better understood.

Concerning exact controllability, local and almost global (between eigenstates) results for 1D models were obtained by Beauchard [8, 9] and Coron and Beauchard [11], respectively. In [12], Beauchard and Laurent proposed important simplifications of the proofs and dealt with nonlinear Schrödinger and wave equations with bilinear controls, but in simpler configurations than in the present article. In [22], Coron proved that a positive minimal time may be required for the local controllability of the 1D model. This subject was studied further by Beauchard and Morancey [14], and by Beauchard for 1D wave equations [10]. Exact controllability has also been studied in infinite time by Nersesyan and Nersisian [47, 48].

As for approximate controllability, Mirrahimi and Beauchard [13] proved global approximate controllability in infinite time for a 1D model, and Mirrahimi obtained a similar result for equations with continuous spectrum [38]. Using adiabatic theory and intersection of eigenvalues in the space of controls, Boscain and Adami proved approximate controllability in finite time for particular models [2]. Approximate controllability, in finite time, for more general models, has been studied by 3 teams, using different tools: Boscain, Chambrion, Mason, Sigalotti [21, 46, 16], used geometric control methods; Nersesyan [40, 41] used feedback controls and variational methods; and Ervedoza and Puel [26] considered a simplified model.

Moreover, optimal control problems have been investigated for Schrödinger equations with a nonlinearity of Hartree type by Baudouin, Kavian, Puel [5, 6] and by Cances, Le Bris, Pilot [25]. Baudouin and Salomon studied an algorithm for the computation of optimal controls [7]. The idea of "finite controllability of infinite-dimensional systems" was introduced by Bloch, Brockett, and Rangan [15]. Finally, we mention that the somewhat related problem of bilinear wave equations was considered by Khapalov [35, 34, 33], who proves global approximate controllability to nonnegative equilibrium states.

1.6 Notation

If X is a normed vector space, $x \in X$ and $R > 0$, $B_X(x,R) := \{y \in X; \|x-y\| < R\}$ denote the open ball with radius R and $\overline{B}_X(x,R) := \{y \in X; \|x-y\| \leqslant R\}$ denotes the closed ball with radius R. Implicitly, functions take complex values, thus we write, for instance $H^1_0(0,1)$ instead of $H^1_0((0,1),\mathbb{C})$. Otherwise we specify it and write, for example $L^2((0,T),\mathbb{R})$, $L^2((0,1),\mathbb{C}^2)$, etc. We denote by $\langle .,. \rangle$ the (complex valued) scalar product in $L^2((0,1),\mathbb{C}^2)$

$$\langle U, V\rangle = \left\langle \begin{pmatrix} U^{(1)} \\ U^{(2)} \end{pmatrix}, \begin{pmatrix} V^{(1)} \\ V^{(2)} \end{pmatrix} \right\rangle = \int_0^1 \left[U^{(1)}(x)\overline{V^{(1)}(x)} + U^{(2)}(x)\overline{V^{(2)}(x)} \right] dx, \tag{15}$$

and the (complex valued) scalar product in $L^2((0,1),\mathbb{C})$

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}dx.$$

[2]For (partial) reviews of (linear and bilinear) control of Schrödinger equations, see for example [49, 31, 23]. On the even broader subject of quantum control, several review papers and monographs are available; for a recent survey, see e.g. [17] and the literature (680 references!) therein.

When the symbols '$\pm$' (resp. '$\mp$') are used, the upper symbol '+' (resp. '$-$') refers to the focusing case, while the lower symbol '$-$' (resp. '+') refers to the defocussing one. This convention holds in all the article, with only one exception explained in Remark 14.

2 Ground states

In this brief section we establish existence, uniqueness and some important properties of the positive solutions ϕ_μ of (13). Proofs will be provided in Appendix A.

Proposition 4 *For every $\mu \in (\mp\pi^2, +\infty)$, there exists a unique positive solution $\phi_\mu \in H^3_{(0)}((0,1),\mathbb{R})$ of (13). Moreover, the map $\mu \in (\mp\pi^2, +\infty) \to \phi_\mu \in L^2(0,1)$ is analytic and*

$$\langle \partial_\mu \phi_\mu, \phi_\mu \rangle > 0, \quad \forall \mu \in (\mp\pi^2, +\infty), \tag{16}$$

$$\|\phi_\mu\|_{L^\infty(0,1)} \underset{\mu \to \mp\pi^2}{\longrightarrow} 0. \tag{17}$$

Remark 5 *ϕ_μ is actually a smooth function (of x), but this property will not be used in this article.*

Remark 6 *Property (16) is known in the literature as "convexity condition" or "Vakhitov-Kolokolov condition" or "slope condition"; it plays an important rôle in the stability of solitary-wave solutions.*

3 Well posedness

Proposition 7 *Let $\mu \in (\mp\pi^2, +\infty)$, $T > 0$, $u \in \dot{H}^1_0((0,T),\mathbb{R})$. There exists a unique (classical) solution $\psi \in C^0([0,T], H^3 \cap H^1_0) \cap C^1([0,T], H^1_0)$ of (7)(14). Moreover $\psi(T) \in H^3_{(0)}(0,1)$ and*

$$\|\psi(t)\|_{L^2(0,1)} = \|\phi_\mu\|_{L^2(0,1)} e^{\frac{1}{2}\int_0^t u(s)ds}, \quad \forall t \in [0,T].$$

This statement will be proved by working on the auxiliary system

$$\begin{cases} i\partial_t \xi = -\partial_x^2 \xi - w(t)|\xi|^2\xi + v(t)x^2\xi, & x \in (0,1), t \in (0,T), \\ \xi(t,0) = \xi(t,1) = 0, & t \in (0,T), \end{cases} \tag{18}$$

with

$$w(t) := \pm e^{\int_0^t u} \qquad \text{and} \qquad v(t) := \frac{1}{4}(\dot{u} - u^2)(t),$$

that results from (7) and the relation

$$\psi(t,x) = \xi(t,x) e^{\frac{i}{4}u(t)x^2 + \frac{1}{2}\int_0^t u(s)ds}. \tag{19}$$

The following proposition ensures the local (in time) well posedness of the associated Cauchy-problem when v is small enough in L^2.

Proposition 8 *Let $R_0 > 0$ and $r > 0$. There exists $T = T(R_0, r) > 0$ and $\delta > 0$ such that, for every $\xi_0 \in H^3_{(0)}(0,1)$ with $\|\xi_0\|_{H^3_{(0)}} < R_0$, $w \in L^\infty((0,T),\mathbb{R})$ with $\|w\|_{L^\infty(0,T)} < r$, and $v \in L^2((0,T),\mathbb{R})$ with $\|v\|_{L^2(0,T)} < \delta$, there exists a unique (classical) solution $\xi \in C^0([0,T], H^3_{(0)}) \cap C^1([0,T], H^1_0)$ of the system (18) with the initial condition*

$$\xi(0,x) = \xi_0(x), \quad x \in (0,1). \tag{20}$$

Moreover $\|\xi(t)\|_{L^2(0,1)} = \|\xi_0\|_{L^2}$, $\forall t \in [0,T]$.

The following technical result, proved in [12, Lemma 1], will be used in the proof of Proposition 8.

Lemma 9 *Let $T>0$ and $f\in L^2((0,T),H^3\cap H^1_0(0,1))$. The function $G:t\mapsto\int_0^t e^{iAs}f(s)ds$ belongs to $C^0([0,T],H^3_{(0)}(0,1))$ and*

$$\|G\|_{L^\infty((0,T),H^3_{(0)})}\leqslant c_1(T)\|f\|_{L^2((0,T),H^3\cap H^1_0)} \tag{21}$$

where the constants $c_1(T)$ are uniformly bounded for T lying in bounded intervals.

Proof of Proposition 8: Let c_1 be the constant of Lemma 9 associated to the value $T=1$. We introduce constants $c_2,c_2',c_3>0$ such that, for every $z,\tilde{z}\in H^3_{(0)}(0,1)$,

$$\begin{gathered}\||z|^2z\|_{H^3_{(0)}}\leqslant c_2\|z\|^3_{H^3_{(0)}},\\ \||z|^2z-|\tilde{z}|^2\tilde{z}\|_{H^3_{(0)}}\leqslant c_2'\|z-\tilde{z}\|_{H^3_{(0)}}\max\{\|z\|^2_{H^3_{(0)}};\|\tilde{z}\|^2_{H^3_{(0)}}\},\\ \|x^2z\|_{H^3}\leqslant c_3\|z\|_{H^3_{(0)}}.\end{gathered} \tag{22}$$

We define

$$R:=3R_0,\qquad \delta:=\frac{1}{3c_1c_3},\qquad\text{and}\qquad T=T(R_0,r):=\min\left\{1;\frac{1}{3c_2rR^2};\frac{1}{2rc_2'R^2}\right\}. \tag{23}$$

Let $v\in L^2((0,T),\mathbb{R})$ with $\|v\|_{L^2}<\delta$ and $w\in L^\infty((0,T),\mathbb{R})$ with $\|w\|_{L^\infty}<r$. We introduce the map

$$\left|\begin{array}{lccc}F: & \overline{B}_{C^0([0,T],H^3_{(0)})}(0,R) & \to & C^0([0,T],H^3_{(0)})\\ & \xi & \to & F(\xi)\end{array}\right.$$

where

$$F(\xi)(t)=e^{-iAt}\xi_0-i\int_0^t e^{-iA(t-s)}\Big[-w(s)|\xi|^2\xi(s)+v(s)x^2\xi(s)\Big]ds,\forall t\in[0,T].$$

Lemma 9 proves that F takes values in $C^0([0,T],H^3_{(0)})$.

First step: We prove that F maps $\overline{B}_{C^0([0,T],H^3_{(0)})}(0,R)$ into itself. Using (23), we get, for every $t\in[0,T]$,

$$\begin{gathered}\|e^{-iAt}\xi_0\|_{H^3_{(0)}}=\|\xi_0\|_{H^3_{(0)}}<R_0=\frac{R}{3},\\ \left\|\int_0^t e^{-iA(t-s)}w(s)|\xi|^2\xi(s)ds\right\|_{H^3_{(0)}}\leqslant r\int_0^t\||\xi|^2\xi(s)\|_{H^3_{(0)}}ds\leqslant rTc_2R^3\leqslant\frac{R}{3}.\end{gathered}$$

By Lemma 9 and (23) we also have, for every $t\in[0,T]$,

$$\left\|\int_0^t e^{-iA(t-s)}v(s)x^2\xi(s)ds\right\|_{H^3_{(0)}}\leqslant c_1\|v\|_{L^2(0,t)}\|x^2\xi\|_{L^\infty((0,t),H^3)}\leqslant c_1c_3\|v\|_{L^2(0,T)}R<\frac{R}{3}.$$

Second step: We prove that F is a contraction of $\overline{B}_{C^0([0,T],H^3_{(0)})}(0,R)$. Working as in the first step, we get, for any $\xi_1,\xi_2\in\overline{B}_{C^0([0,T],H^3_{(0)})}(0,R)$ the following estimates

$$\begin{aligned}\left\|\int_0^t e^{-iA(t-s)}w(s)[|\xi_1|^2\xi_1(s)-|\xi_2|^2\xi_2(s)]ds\right\|_{H^3_{(0)}}&\leqslant Trc_2'\|\xi_1-\xi_2\|_{L^\infty(H^3_{(0)})}R^2\\ &\leqslant\frac{\|\xi_1-\xi_2\|_{L^\infty(H^3_{(0)})}}{2},\\ \left\|\int_0^t e^{-iA(t-s)}v(s)x^2(\xi_1-\xi_2)(s)ds\right\|_{H^3_{(0)}}&\leqslant c_1c_3\|v\|_{L^2(0,T)}\|\xi_1-\xi_2\|_{L^\infty(H^3_{(0)})}\\ &\leqslant\frac{\|\xi_1-\xi_2\|_{L^\infty(H^3_{(0)})}}{3},\end{aligned}$$

where $L^\infty(H^3_{(0)}) = L^\infty((0,T),H^3_{(0)})$.

Third step: Conclusion. By applying the Banach fixed point theorem to the map F, we get a function $\xi \in \overline{B}_{C^0([0,T],H^3_{(0)})}(0,R)$ such that $F(\xi)=\xi$. From this equality, we deduce that $\xi \in C^1([0,T],H^1_0)$ and that the first equality of (18) holds in $H^1_0(0,1)$ for every $t\in[0,T]$. In particular, ξ is a classical solution of the equation. $\square$

The following proposition ensures that maximal solutions of (18) are global in time.

Proposition 10 *Let $T>0$, $\xi_0 \in H^3_{(0)}(0,1)$, $v\in L^2((0,T),\mathbb{R})$ and $w\in H^1((0,T),\mathbb{R})$. There exists a unique (classical) solution $\xi \in C^0([0,T],H^3_{(0)})\cap C^1([0,T],H^1_0)$ of the system (18)(20). There exists $C=C(\|\xi_0\|_{H^3_{(0)}},\|v\|_{L^2(0,T)},\|w\|_{H^1(0,T)})>0$ such that*

$$\|\xi\|_{L^\infty((0,T),H^3_{(0)})} \leqslant C.$$

Moreover $\|\xi(t)\|_{L^2}=\|\xi_0\|_{L^2}$, $\forall t\in[0,T]$.

Then, Proposition 7 follows from Proposition 10 and the change of variable (19).

Proof of Proposition 10: We extend v by zero and w by $w(T)$ on $(T,+\infty)$. Our goal is to prove the existence and uniqueness of a solution $\xi\in C^0([0,+\infty),H^3_{(0)})\cap C^1([0,+\infty),H^1_0)$ of (18)(20).

First step: Maximal solution. By Proposition 8, there exists a unique local (in time) solution $\xi\in C^0([0,T_1],H^3_{(0)})\cap C^1([0,T_1],H^1_0)$ of (18)(20), for some time $T_1>0$. The uniqueness of Proposition 8 and Zorn Lemma imply the existence of a unique maximal solution $\xi\in C^0([0,T^*),H^3_{(0)})\cap C^1([0,T^*),H^1_0)$ of (18)(20), for some time $T^*\in(0,+\infty]$. Now, we prove by contradiction that $T^*=\infty$. We assume that $T^*<+\infty$.

Second step: We prove that $\xi(t)$ is bounded in $H^1_0(0,1)$ uniformly with respect to $t\in[0,T^*)$. We recall that $\xi\in C^1([0,T^*),H^1_0)$, and the first equality of (18) holds in $H^1_0(0,1)$ for every $t\in[0,T]$. Thus, the function

$$J(t) := \int_0^1 \left(\frac{1}{2}|\partial_x\xi(t,x)|^2 - \frac{w(t)}{4}|\xi(t,x)|^4\right)dx$$

satisfies

$$\frac{dJ}{dt}(t) = 2v(t)\mathrm{Im}\left(\int_0^1 x\overline{\partial_x\xi(t,x)}\xi(t,x)dx\right) - \frac{\dot{w}(t)}{4}\|\xi(t)\|^4_{L^4}. \tag{24}$$

We also recall the existence of a constant $\mathcal{C}>0$ such that (Galiardo-Nirenberg inequality [27, p. 147])

$$\|f\|_{L^4(0,1)} \leqslant \mathcal{C}\|f\|^{3/4}_{L^2(0,1)}\|\partial_x f\|^{1/4}_{L^2(0,1)}, \quad \forall f\in H^1_0(0,1).$$

For every $t\in[0,T^*)$, we have

$$\begin{aligned} -\tfrac{w(t)}{4}\|\xi(t)\|^4_{L^4} &\leqslant \tfrac{\mathcal{C}}{4}\|w\|_{L^\infty(0,T^*)}\|\xi(t)\|^3_{L^2}\|\partial_x\xi(t)\|_{L^2} \\ &\leqslant \tfrac{1}{4}\|\partial_x\xi(t)\|^2_{L^2} + \tfrac{\mathcal{C}^2}{16}\|w\|^2_{L^\infty(0,T)}\|\xi_0\|^6_{L^2} \end{aligned}$$

thus

$$J(t) \geqslant \frac{1}{4}\|\partial_x\xi(t)\|^2_{L^2} - \frac{\mathcal{C}^2}{16}\|w\|^2_{L^\infty(0,T^*)}\|\xi_0\|^6_{L^2}, \quad \forall t\in[0,T^*). \tag{25}$$

We deduce that

$$\begin{aligned} 2v(t)\mathrm{Im}\left(\textstyle\int_0^1 x\overline{\partial_x\xi(t,x)}\xi(t,x)dx\right) &\leqslant 2|v(t)|\|\partial_x\xi(t)\|_{L^2} \\ &\leqslant 4v(t)^2 + \tfrac{1}{4}\|\partial_x\xi(t)\|^2_{L^2} \\ &\leqslant 4v(t)^2 + J(t) + \tfrac{\mathcal{C}^2}{16}\|w\|^2_{L^\infty(0,T^*)}\|\xi_0\|^6_{L^2} \end{aligned} \tag{26}$$

and

$$
\begin{array}{rl}
-\frac{\dot{w}(t)}{4}\|\xi(t)\|_{L^4}^4 & \leqslant \frac{\mathcal{C}}{4}|\dot{w}(t)|\|\partial_x\xi(t)\|_{L^2} \\
& \leqslant \frac{\mathcal{C}^2}{16}\dot{w}(t)^2 + \frac{1}{4}\|\partial_x\xi(t)\|_{L^2}^2 \\
& \leqslant \frac{\mathcal{C}^2}{16}\dot{w}(t)^2 + J(t) + \frac{\mathcal{C}^2}{16}\|w\|_{L^\infty(0,T^*)}^2\|\xi_0\|_{L^2}^6.
\end{array} \tag{27}
$$

From (24), (26), (27) and Gronwall lemma, we get

$$
J(t) \leqslant \left(J(0) + \int_0^t \left(4v(s)^2 + \frac{\mathcal{C}^2}{16}[\dot{w}(s)^2 + 2\|w\|_{L^\infty(0,T^*)}^2\|\xi_0\|_{L^2}^6]\right)ds\right)e^{2t}, \forall t \in [0,T^*).
$$

Thus, J is bounded uniformly with respect to $t \in [0,T^*)$, and so is $\|\xi(t)\|_{H^1}$ (see (25)).

Third step: We prove that $\xi(t)$ is bounded in $H^3_{(0)}(0,1)$ uniformly with respect to $t \in [0,T^*)$. First, we recall the existence of a constant $\mathcal{C}$ such that

$$
\||\xi|^2\xi\|_{H^3_{(0)}} \leqslant \mathcal{C}\|\xi\|_{H^3_{(0)}}\|\xi\|_{H^1_0}^2, \quad \forall \xi \in H^3_{(0)}(0,1).
$$

This follows from the explicit expression of $\partial_x^3[|\xi|^2\xi]$ and the Galiardo-Nirenberg inequality. From the relation $\xi = F(\xi)$ in $C^0([0,T],H^3_{(0)})$ and Lemma 9, we get, for every $t \in [0,T^*)$,

$$
\|\xi(t)\|_{H^3_{(0)}} \leqslant \|\xi_0\|_{H^3_{(0)}} + \int_0^t |w(s)|\mathcal{C}\|\xi\|_{L^\infty(H^1)}^2\|\xi(s)\|_{H^3_{(0)}}ds + c_1(T^*)\left(\int_0^t |v(s)|^2c_3^2\|\xi(s)\|_{H^3_{(0)}}^2ds\right)^{1/2}
$$

(see (22) for the definition of c_3). Using Cauchy-Schwarz inequality, we get

$$
\begin{array}{rl}
\|\xi(t)\|_{H^3_{(0)}}^2 \leqslant & 3\|\xi_0\|_{H^3_{(0)}}^2 + 3t\int_0^t |w(s)|^2\mathcal{C}^2\|\xi\|_{L^\infty(H^1)}^4\|\xi(s)\|_{H^3_{(0)}}^2 ds \\
& +3c_1(T^*)^2\int_0^t |v(s)|^2c_3^2\|\xi(s)\|_{H^3_{(0)}}^2 ds.
\end{array}
$$

Then Gronwall lemma proves that $\xi(t)$ is bounded in $H^3_{(0)}$ uniformly with respect to $t \in [0,T^*]$.

Fourth step: Conclusion. From the relation $\xi(t) = F(\xi)(t)$ and the third step, $\xi(t)$ satisfies the Cauchy-criterion in $H^3_{(0)}(0,1)$ when $[t \to T^*]$. Thus the maximal solution may be extended after T^*, which is a contradiction. Therefore $T^* = +\infty$. $\Box$

4 C^1-regularity of the end-point map

By Proposition 7, we can consider, for any $T > 0$ and $\mu \in (\mp\pi^2, +\infty)$ the end point map

$$
\left|\begin{array}{cccc}
\Theta_{T,\mu}: & \dot{H}^1_0((0,T),\mathbb{R}) & \to & H^3_{(0)}(0,1) \cap \mathcal{S}_{\|\phi_\mu\|_{L^2}} \\
 & u & \mapsto & \psi(T)
\end{array}\right.
$$

where ψ is the solution of (7)(14) and $\mathcal{S}_{\|\phi_\mu\|_{L^2}}$ is the $L^2((0,1),\mathbb{C})$-sphere with radius $\|\phi_\mu\|_{L^2}$. The goal of this section is the proof of the C^1-regularity of $\Theta_{T,\mu}$.

Proposition 11 *Let $\mu \in (\mp\pi^2, +\infty)$ and $T > 0$. The map $\Theta_{T,\mu}$ is C^1, moreover, for every $u, U \in \dot{H}^1_0((0,T),\mathbb{R})$, we have $d\Theta_{T,\mu}(u).U = \Psi(T)$ where Ψ is the solution of the linearized system*

$$
\left\{\begin{array}{ll}
i\partial_t\Psi = -\partial_x^2\Psi \mp [2|\psi|^2\Psi + \psi^2\overline{\Psi}] + iU(t)\partial_x[x\psi], & x \in (0,1), t \in (0,T), \\
\Psi(t,0) = \Psi(t,1) = 0, & t \in (0,T). \\
\Psi(0,x) = 0, & x \in (0,1).
\end{array}\right. \tag{28}
$$

and ψ is the solution of (7)(14).

This proposition will be proved by working first on the auxiliary system (18).

4.1 For the auxiliary system (18)

For $\mu \in (\mp\pi^2, +\infty)$, we introduce the end-point map of the auxiliary system

$$\left| \begin{array}{llll} \Omega_{T,\mu}: & L^2 \times H^1((0,T),\mathbb{R}) & \to & H^3_{(0)}(0,1) \cap \mathcal{S}_{\|\phi_\mu\|_{L^2}} \\ & (v,w) & \mapsto & \xi(T) \end{array} \right.$$

where ξ is the solution (18) with the initial condition

$$\xi(0,x) = \phi_\mu(x), \quad x \in (0,1). \tag{29}$$

Proposition 12 *Let $T > 0$. The map $\Omega_{T,\mu}$ is C^1, moreover, for every $(v,w) \in L^2 \times H^1((0,T),\mathbb{R})$, we have $d\Omega_{T,\mu}(v,w).(V,W) = \zeta(T)$ where ζ is the solution of the linearized system*

$$\left\{ \begin{array}{ll} i\partial_t \zeta = -\partial_x^2 \zeta - w(t)[2|\xi|^2\zeta + \xi^2\overline{\zeta}] + v(t)x^2\zeta - W(t)|\xi|^2\xi + V(t)x^2\xi, & x \in (0,1), t \in (0,T), \\ \zeta(t,0) = \zeta(t,1) = 0, & t \in (0,T). \\ \zeta(0,x) = 0, & x \in (0,1). \end{array} \right. \tag{30}$$

and ξ is the solution of (18)(29).

Proof of Proposition 12:

First step: Well posedness of (30). Let $(v,w),(V,W) \in L^2 \times H^1((0,T),\mathbb{R})$ and ξ be the solution of (18)(29). The well posedness of (30) may be proved with a fixed point argument in $C^0([0,T_1],H^3_{(0)})$ under a smallness assumption on $\|w\|_{L^1(0,T_1)}$ and $\|v\|_{L^2(0,T_1)}$ for the map to be contracting. Then, iterating this argument on a finite number of intervals $[0,T_1]$, $[T_1,T_2]$,... we get the well posedness of (30) on the whole interval $[0,T]$.

Second step: Local Lipschitz regularity of $\Omega_{T,\mu}$. Let $(v,w) \in L^2 \times H^1((0,T),\mathbb{R})$ and ξ be the solution of (18)(29). Let $(V,W) \in L^2 \times H^1((0,T),\mathbb{R})$ with $\|(V,W)\|_{L^2\times H^1(0,T)} \leqslant 1$ and $\widetilde{\xi}$ be the solution of

$$\left\{ \begin{array}{ll} i\partial_t \widetilde{\xi} = -\partial_x^2 \widetilde{\xi} - (w+W)(t)|\widetilde{\xi}|^2\widetilde{\xi} + (v+V)(t)x^2\widetilde{\xi}, & x \in (0,1), t \in (0,T), \\ \widetilde{\xi}(t,0) = \widetilde{\xi}(t,1) = 0, & t \in (0,T), \\ \widetilde{\xi}(0,x) = \phi_\mu(x), & x \in (0,1). \end{array} \right.$$

We claim that there exists a constant $C_1 = C_1(\|v\|_{L^2}, \|w\|_{H^1}) > 0$ (independent of (V,W)) such that

$$\|\widetilde{\xi} - \xi\|_{L^\infty(H^3_0)} \leqslant C_1 \|(V,W)\|_{L^2\times H^1}. \tag{31}$$

By Proposition 10, there exists $R = R(\|v\|_{L^2}, \|w\|_{H^1}) > 0$ (independent of V and W) such that

$$\|\xi\|_{L^\infty(H^3_{(0)})}, \|\widetilde{\xi}\|_{L^\infty(H^3_{(0)})} \leqslant R. \tag{32}$$

Thus, there exists $C_2 = C_2(R) > 0$ such that

$$\||\widetilde{\xi}|^2\widetilde{\xi} - |\xi|^2\xi\|_{L^\infty(H^3_{(0)})} \leqslant C_2 \|\widetilde{\xi} - \xi\|_{L^\infty(H^3_{(0)})}.$$

From the relation

$$(\widetilde{\xi} - \xi)(t) = -i\int_0^t e^{-iAs}\left[-w[|\widetilde{\xi}|^2\widetilde{\xi} - |\xi|^2\xi] - W|\widetilde{\xi}|^2\widetilde{\xi} + vx^2(\widetilde{\xi} - \xi) + Vx^2\widetilde{\xi}\right](s)ds,$$

Lemma 9 and (22), we get

$$\begin{array}{rl} \|(\widetilde{\xi} - \xi)(t)\|_{H^3_{(0)}} \leqslant & \int_0^t \left(|w(s)|C_2\|(\widetilde{\xi} - \xi)(s)\|_{H^3_{(0)}} + |W(s)|c_2R^3\right)ds \\ & + c_1(T)\left(\int_0^t \left[|v(s)|^2 c_3^2 \|(\widetilde{\xi} - \xi)(s)\|^2_{H^3_{(0)}} + |V(s)|^2 c_3^2 R^2\right]ds\right)^{1/2}. \end{array}$$

Thus,

$$\begin{array}{rl}\|(\widetilde{\xi}-\xi)(t)\|^2_{H^3_{(0)}} \leqslant & 4t\int_0^t \left(|w(s)|^2 C_2^2 \|(\widetilde{\xi}-\xi)(s)\|^2_{H^3_{(0)}} + |W(s)|^2 c_2^2 R^6 \right) ds \\ & +2c_1(T)^2 \int_0^t \left[|v(s)|^2 c_3^2 \|(\widetilde{\xi}-\xi)(s)\|^2_{H^3_{(0)}} + |V(s)|^2 c_3^2 R^2 \right] ds \end{array}$$

and we get (31) thanks to Gronwall lemma.

Third step: Existence of a constant $C = C(\|v\|_{L^2}, \|w\|_{H^1}) > 0$ such that

$$\|\widetilde{\xi}-\xi-\zeta\|_{L^\infty(H^3_{(0)})} \leqslant C\|(V,W)\|^2_{L^2\times H^1}, \text{ when } \|(V,W)\|_{L^2\times H^1} \leqslant 1.$$

Thanks to (32), there exists a constant $C_3 = C_3(R) > 0$ such that

$$\||\widetilde{\xi}|^2\widetilde{\xi} - |\xi|^2\xi - 2|\xi|^2(\widetilde{\xi}-\xi) - \xi^2\overline{(\widetilde{\xi}-\xi)}\|_{L^\infty(H^3_{(0)})} \leqslant C_3\|\widetilde{\xi}-\xi\|^2_{L^\infty(H^3_{(0)})}.$$

Let $\Delta := \widetilde{\xi}-\xi-\zeta$. From the relation

$$\begin{array}{rl}\Delta(t) = -i\int_0^t e^{-iAs}[& -w(s)[|\widetilde{\xi}|^2\widetilde{\xi}(s) - |\xi|^2\xi(s) - 2|\xi|^2(\widetilde{\xi}-\xi)(s) - \xi^2\overline{(\widetilde{\xi}-\xi)}(s)] \\ & -w(s)[2|\xi|^2(\widetilde{\xi}-\xi-\zeta) + \xi^2\overline{(\widetilde{\xi}-\xi-\zeta)} \\ & -W(s)[|\widetilde{\xi}|^2\widetilde{\xi}(s) - |\xi|^2\xi(s)] \\ & +v(s)x^2\Delta(s) + V(s)x^2(\widetilde{\xi}-\xi)(s)\ ]\, ds \end{array}$$

we deduce that

$$\begin{array}{rl}\|\Delta(t)\|_{H^3_0} \leqslant & \int_0^t |w(s)|\left(C_3 C_1^2 \|(V,W)\|^2_{L^2\times H^1} + 3R^2\|\Delta(s)\|_{H^3_{(0)}} \right) ds \\ & + \int_0^t |W(s)| C_2 C_1 \|(V,W)\|_{L^2\times H^1} ds \\ & + \left(\int_0^t \left[|v(s)|^2 c_3^2 \|\Delta(s)\|^2_{H^3_{(0)}} + |V(s)|^2 c_3^2 C_1^2 \|(V,W)\|^2_{L^2\times H^1} \right] ds \right)^{1/2}. \end{array}$$

We conclude the proof by taking the square of this inequality and applying Gronwall lemma. □

4.2 For the system (7)

We now prove Proposition 11. First, we recall that, for every $u \in \dot{H}^1_0((0,T),\mathbb{R})$, $\Theta_{T,\mu}(u) = \Omega_{T,\mu}(v,w)$, where $w(t) := \pm e^{\int_0^t u}$ and $v(t) := \frac{(\dot{u}-u^2)(t)}{4}$. Thus $\Theta_{T,\mu}$ is C^1 and

$$d\Theta_{T,\mu}(u).U = d\Omega_{T,\mu}(v,w).(V,W) \quad \text{where} \quad V := \frac{\dot{U}-2uU}{4} \text{ and } W := \pm\left(\int_0^t U\right) e^{\int_0^t u}.$$

This gives the conclusion because

$$\Psi(t,x) = \left[\zeta(t,x) + \left(\frac{i}{4}U(t)x^2 + \frac{1}{2}\int_0^t U(s)ds\right)\xi(t,x)\right] e^{\frac{i}{4}u(t)x^2 + \frac{1}{2}\int_0^t u(s)ds}.$$

5 Spectral analysis and consequences

In this section, we are interested in the linearized system around the nonlinear trajectory $(\psi_\mu(t,x) = \phi_\mu(x)e^{\pm i\mu t}, u = 0)$ where ϕ_μ is defined by (13), for $\mu \in (\mp\pi^2, +\infty)$,

$$\left\{\begin{array}{ll} i\partial_t\Psi = -\partial_x^2\Psi \mp [2|\psi_\mu|^2\Psi + \psi_\mu^2\overline{\Psi}] + iU(t)\partial_x[x\psi_\mu], & x \in (0,1), t \in (0,T), \\ \Psi(t,0) = \Psi(t,1) = 0, & t \in (0,T), \\ \Psi(0,x) = 0, & x \in (0,1). \end{array}\right.$$

As usual, the time dependence of the second term in the right hand side is eliminated by the transformation

$$\Psi(t,x) = \widetilde{\Psi}(t,x)e^{\pm i\mu t}$$

which leads to

$$\begin{cases} i\partial_t \widetilde{\Psi} = -\partial_x^2 \widetilde{\Psi} \pm \mu \widetilde{\Psi} \mp [2\phi_\mu^2 \widetilde{\Psi} + \phi_\mu^2 \overline{\widetilde{\Psi}}] + iU(t)(x\phi_\mu)', & x \in (0,1), t \in (0,T), \\ \widetilde{\Psi}(t,0) = \widetilde{\Psi}(t,1) = 0, & t \in (0,T), \\ \widetilde{\Psi}(0,x) = 0, & x \in (0,1). \end{cases} \tag{33}$$

In this section, we will work with the real (2×2)-system arising from this equation, by decomposition in real and imaginary parts. Consider the matrix operator

$$\mathcal{L}_\mu := \begin{pmatrix} 0 & L_\mu^- \\ -L_\mu^+ & 0 \end{pmatrix} \quad \text{where} \quad \begin{cases} L_\mu^- := -\Delta \pm \mu \mp \phi_\mu^2, \\ L_\mu^+ := -\Delta \pm \mu \mp 3\phi_\mu^2. \end{cases} \tag{34}$$

The previous equation takes the form

$$\begin{cases} \partial_t Z = \mathcal{L}_\mu Z + U(t)(x\phi_\mu)' \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ Z(t,0) = Z(t,1) = 0, \\ Z(0,x) = 0, \end{cases} \tag{35}$$

where

$$Z(t,x) := \begin{pmatrix} \mathrm{Re}[\widetilde{\Psi}(t,x)] \\ \mathrm{Im}[\widetilde{\Psi}(t,x)] \end{pmatrix}.$$

For convenience, we also define

$$\mathcal{L}_{\mp\pi^2} := \begin{pmatrix} 0 & -\Delta - \pi^2 \\ \Delta + \pi^2 & 0 \end{pmatrix}, \qquad \phi_{\mp\pi^2} := 0. \tag{36}$$

The goal of this section is to establish the spectral properties of the operators $\mathcal{L}_\mu$ needed in the proof of the controllability of the linear system (33) in Section 6.

5.1 Auxiliary operators

It will be convenient to employ a similarity transformation (see [42, (12.15)]). Let

$$J := \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}. \tag{37}$$

Then, for any $\mu \in [\mp\pi^2, +\infty)$, we have

$$i\mathcal{L}_\mu = J^{-1}\mathcal{M}_\mu J \text{ where } \mathcal{M}_\mu := \begin{pmatrix} -\Delta & 0 \\ 0 & \Delta \end{pmatrix} + \begin{pmatrix} \pm\mu \mp 2\phi_\mu^2 & \mp\phi_\mu^2 \\ \pm\phi_\mu^2 & \mp\mu \pm 2\phi_\mu^2 \end{pmatrix} =: \mathcal{D} + \widetilde{\mathcal{M}}_\mu. \tag{38}$$

Note that $J^* = 2J^{-1}$ and so

$$\mathrm{Sp}(\mathcal{L}_\mu) = i\,\mathrm{Sp}(\mathcal{M}_\mu), \quad \forall \mu \in [\mp\pi^2, +\infty).$$

5.2 Basic spectral properties

In this section, we recall basic spectral properties of the operators $\mathcal{L}_\mu$ and $\mathcal{M}_\mu$. For this article to be self-contained, we propose proofs in Appendix B.

Proposition 13 *Let $\mu \in [\mp\pi^2, +\infty)$.*

(i) The spectrum of $\mathcal{M}_\mu$ and $\mathcal{M}_\mu^*$ is purely discrete and the systems of eigenvectors and generalized eigenvectors for $\mathcal{M}_\mu$ and $\mathcal{M}_\mu^*$ (and hence for $\mathcal{L}_\mu$ and $\mathcal{L}_\mu^*$) form Schauder bases for $L^2((0,1),\mathbb{C}^2)$.

(ii) All non-zero eigenvalues of $\mathcal{L}_\mu$ are purely imaginary: $Sp(\mathcal{L}_\mu) = \{\pm i\beta_{n,\mu}; n\in\mathbb{N}\}$ where $(\beta_{n,\mu})_{n\in\mathbb{N}} \subset [0,+\infty)^{\mathbb{N}}$ is non decreasing (here, multiple eigenvalues are repeated).

(iii) There exists $n_* = n_*(\mu)\in\mathbb{N}$ and $C = C(\mu) > 0$ such that

$$|\beta_{n,\mu} - (n+n^*)^2\pi^2| \le C, \quad \forall n\in\mathbb{N}. \tag{39}$$

(iv) The function $\mu\mapsto\beta_{n,\mu}$ is continuous for every $n\in\mathbb{N}$ and

$$\beta_{n,\mp\pi^2} = [(n+1)^2-1]\pi^2, \quad \forall n\in\mathbb{N}. \tag{40}$$

(v) The multiplicity of the eigenvalues of $\mathcal{L}_\mu$ is at most two. No non-zero eigenvalue possesses a generalized eigenvector.

(vi) The vectors

$$\Phi_0^+ = \begin{pmatrix} 0 \\ \phi_\mu \end{pmatrix} \quad \text{and} \quad \Phi_0^- = \begin{pmatrix} \partial_\mu\phi_\mu \\ 0 \end{pmatrix}$$

satisfy

$$\mathcal{L}_\mu\Phi_0^- = \Phi_0^+, \qquad \mathcal{L}_\mu\Phi_0^+ = 0. \tag{41}$$

Moreover (Φ_0^+,Φ_0^-) is a basis of the generalized null space for $\mathcal{L}_\mu$. The vectors

$$\Psi_0^- = \begin{pmatrix} \phi_\mu \\ 0 \end{pmatrix}, \quad \text{and} \quad \Psi_0^+ = \begin{pmatrix} 0 \\ \partial_\mu\phi_\mu \end{pmatrix} \tag{42}$$

satisfy

$$\mathcal{L}_\mu^*\Psi_0^+ = \Psi_0^-, \qquad \mathcal{L}_\mu^*\Psi_0^- = 0. \tag{43}$$

Moreover, (Ψ_0^+,Ψ_0^-) is a basis of the generalized null space of $\mathcal{L}_\mu^*$.

(vii) Let $(\Phi_n^+)_{n\in\mathbb{N}^*}$ be normalized (see remark 15 below) eigenvectors of $\mathcal{L}_\mu$ associated to the eigenvalues $(+i\beta_{n,\mu})_{n\in\mathbb{N}^*}$ and $\Phi_n^- := \overline{\Phi_n^+}$, then

$$\mathcal{L}_\mu\Phi_n^\pm = \pm i\beta_{n,\mu}\Phi_n^\pm, \quad \forall n\in\mathbb{N}^*.$$

Let $(\Psi_n^+)_{n\in\mathbb{N}^*}$ be normalized eigenvectors of $\mathcal{L}_\mu^*$ associated to the eigenvalues $(-i\beta_{n,\mu})_{n\in\mathbb{N}^*}$ and $\Psi_n^- := \overline{\Psi_n^+}$, then

$$\mathcal{L}_\mu^*\Psi_n^\pm = \mp i\beta_{n,\mu}\Psi_n^\pm, \quad \forall n\in\mathbb{N}^*.$$

Moreover, if all non zero eigenvalue of $\mathcal{L}_\mu$ is simple then

$$\langle\Phi_m^\sigma,\Psi_n^\tau\rangle = \delta_{m,n}^{\sigma,\tau} := \begin{cases} 1, & m=n \text{ and } \sigma=\tau, \\ 0, & \text{otherwise,} \end{cases} \qquad \forall m,n\in\mathbb{N}, \sigma,\tau\in\{+,-\} \tag{44}$$

where the inner product is defined by (15).

(viii) Let

$$V_n^\mp := J\Phi_n^\pm, \qquad W_n^\mp := J\Psi_n^\pm, \quad \forall n\in\mathbb{N}^*,$$

then,

$$\mathcal{M}_\mu V_n^\pm = \pm\beta_{n,\mu}V_n^\pm, \qquad \mathcal{M}_\mu^* W_n^\pm = \pm\beta_{n,\mu}W_n^\pm, \quad \forall n\in\mathbb{N}^*.$$

Remark 14 *When we use the vectors $\Phi_n^\pm$, $\Psi_n^\pm$, $V_n^\pm$, $W_n^\pm$ the symbols '$\pm$' and '$\mp$' do not refer to a distinction between the focusing and defocussing cases, but to the sign of the associated eigenvalue.*

Remark 15 *Note that in the previous statement, the vectors Φ_n^σ and Ψ_n^σ are defined up to a constant $c_n^\sigma \neq 0$, for every $n \geqslant 1$ and $\sigma \in \{\pm\}$: $\langle c_m^\sigma \Phi_m^\sigma, \Psi_n^\tau / c_n^\tau \rangle = \delta_{m,n}^{\sigma,\tau}$ for every sequence $(c_n^\pm)_{n\in\mathbb{N}^*} \subset \mathbb{R}^*$. The 'normalization' we refer to in the statement* (v) *will be chosen in Proposition 17.*

Remark 16 *We should have written $\Phi_{n,\mu}^\pm$, $\Psi_{n,\mu}^\pm$, $V_{n,\mu}^\pm$, $W_{n,\mu}^\pm$ because these vectors depend on μ. We do not precise μ in subscript in order to simplify the notations.*

5.3 Asymptotics of eigenvectors

In the sequel, we use the $\mathcal{O}$-notation for uniform estimates:

$$f_n(x) = g_n(x) + \mathcal{O}(n^{-\alpha})$$

is to mean that there exists a constant C and functions $R_n(x)$ such that

$$f_n(x) = g_n(x) + R_n(x) n^{-\alpha} \quad \text{and} \quad |R_n(x)| \leq C, \forall x \in (0,1), \forall n \in \mathbb{N}.$$

Proposition 17 *Let $\mu \in (\mp\pi^2, +\infty)$ and $n_* = n_*(\mu) \in \mathbb{N}$ be as in (39). The normalization of $(\Phi_n^\pm, \Psi_n^\pm)$ may be chosen such that*

$$V_n^\pm(x) = 2\sin[(n+n_*)\pi x] e^\pm + \mathcal{O}(1/n), \tag{45a}$$

$$W_n^\pm(x) = 2\sin[(n+n_*)\pi x] e^\pm + \mathcal{O}(1/n), \tag{45b}$$

where $e^+ = \binom{1}{0}$ and $e^- = \binom{0}{1}$.

Proof of Proposition 17: In this proof, we omit the μ in subscripts, in order to simplify the notations, and we deal with the focusing case (the defocussing case may be treated similarly). First, we prove the estimate on

$$V_n^+(x) := \begin{pmatrix} u_n(x) \\ v_n(x) \end{pmatrix}. \tag{46}$$

The equation $\mathcal{M}_\mu V_n^+ = \beta_n V_n^+$ gives

$$u_n'' + (\beta_n - \mu) u_n = -\phi_\mu^2 (2u_n + v_n), \qquad u_n(0) = u_n(1) = 0, \tag{47a}$$

$$v_n'' - (\beta_n + \mu) v_n = -\phi_\mu^2 (u_n + 2v_n), \qquad v_n(0) = v_n(1) = 0. \tag{47b}$$

For n large enough, $(\beta_n + \mu)$ is positive (see (39)), thus $\omega_n := \sqrt{\beta_n + \mu}$ is well defined. From the relations

$$\begin{cases} u_n'' + [(n+n_*)\pi]^2 u_n = f_n(x) := \Big([(n+n_*)\pi]^2 - \beta_n + \mu - 2\phi_\mu^2\Big) u_n - \phi_\mu v_n, \\ u_n(0) = u_n(1) = 0 \end{cases}$$

we deduce that

$$u_n(x) = c \sin[(n+n_*)\pi x] + \frac{1}{(n+n_*)\pi} \int_0^x \sin[(n+n_*)\pi(x-\sigma)] f_n(\sigma) d\sigma \tag{48}$$

for some constant $c \in \mathbb{R}$ that may be taken equal to 2 (see Remark 15). We deduce from (39) that $u_n(x) = 2\sin[(n+n_*)\pi x] + \mathcal{O}(1/n)$. From (47b), we deduce that

$$v_n(x) = -\int_0^1 G_{\omega_n}(x,\sigma) \phi_\mu(\sigma)^2 [u_n(\sigma) + 2v_n(\sigma)] d\sigma, \tag{49}$$

where

$$G_{\omega_n}(x,\sigma) = -\frac{\sinh(\omega_n x)\sinh[\omega_n(1-\sigma)]}{\omega_n \sinh(\omega_n)} + \frac{\sinh[\omega_n(x-\sigma)]}{\omega_n} 1_{\sigma<x}.$$

The function $|G_{\omega_n}(x,\sigma)|$ assumes its maximum on $[0,1]^2$ at the point $(x,\sigma) = \left(\frac{1}{2},\frac{1}{2}\right)$ and its maximum value is given by

$$\left|G_{\omega_n}\left(\tfrac{1}{2},\tfrac{1}{2}\right)\right| = \frac{\sinh^2(\frac{\omega_n}{2})}{\omega_n \sinh(\omega_n)} = \frac{\cosh(\omega_n)-1}{2\omega_n \sinh(\omega_n)} = \mathcal{O}\left(1/\omega_n\right). \tag{50}$$

Thus, (49) and (39) justify that $v_n(x) = \mathcal{O}(1/n)$.

The estimate on V_n^- follows because

$$V_n^- = \begin{pmatrix} \overline{v_n} \\ \overline{u_n} \end{pmatrix}. \tag{51}$$

Working similarly, we get the existence of a constant C_n such that

$$W_n^{\pm}(x) = 2C_n \sin[(n+n_*)\pi x]e^{\pm} + \mathcal{O}\left(1/n\right).$$

Thus,

$$\begin{aligned}
\delta_{\sigma,\tau}^{n,m} &= \langle \Phi_m^{\sigma}, \Psi_n^{\tau}\rangle \\
&= \langle J^{-1}V_m^{\sigma'}, J^{-1}W_n^{\tau'}\rangle \\
&= \tfrac{1}{2}\langle V_m^{\sigma'}, W_n^{\tau'}\rangle \\
&= \tfrac{1}{2}\langle 2\sin[(m+n_*)\pi x]e^{\sigma'} + O\left(\tfrac{1}{m}\right), 2C_n \sin[(n+n_*)\pi x]e^{\tau'} + O\left(\tfrac{1}{n}\right)\rangle \\
&= \delta_{\sigma,\tau} 2C_n \int_0^1 \sin[(m+n_*)\pi x]\sin[(n+n_*)\pi x]dx + O\left(\tfrac{1}{m}+\tfrac{1}{n}\right) \\
&= C_n \delta_{\sigma,\tau}^{n,m} + O\left(\tfrac{1}{m}+\tfrac{1}{n}\right).
\end{aligned}$$

Thus $C_n = 1 + O(1/n)$ when $n \to +\infty$, which gives the conclusion. ☐

Proposition 18 *Let $\mu \in (\mp\pi^2, +\infty)$ and $n_* = n_*(\mu) \in \mathbb{N}$ be as in (39). We denote $V_n^+(x) = \binom{u_n(x)}{v_n(x)}$ and $W_n^+(x) = \binom{w_n(x)}{z_n(x)}$. There exist $\rho_n, \sigma_n, \widetilde{\rho}_n, \widetilde{\sigma_n} \in C^1([0,1],\mathbb{C})$, and $C>0$ such that*

$$\|\rho_n\|_{C^1([0,1])}, \|\widetilde{\rho}_n\|_{C^1([0,1])}, \|\sigma_n\|_{C^1([0,1])}, \|\widetilde{\sigma}_n\|_{C^1([0,1])} \leq C, \quad \forall n \in \mathbb{N}^*, \tag{52}$$

$$\rho_n(0) = \rho_n(1) = \widetilde{\rho}_n(0) = \widetilde{\rho}_n(1) = 0, \quad \forall n \in \mathbb{N}^*, \tag{53}$$

$$u_n(x) = 2\sin[(n+n_*)\pi x] + \frac{\sin[(n+n_*)\pi x]}{(n+n_*)\pi}\rho_n(x) - \frac{\cos[(n+n_*)\pi]}{(n+n_*)\pi}\sigma_n(x) + \mathcal{O}\left(1/n^2\right), \tag{54}$$

$$w_n(x) = 2\sin[(n+n_*)\pi x] + \frac{\sin[(n+n*)\pi x]}{(n+n_*)\pi}\widetilde{\rho}_n(x) - \frac{\cos[(n+n_*)\pi x]}{(n+n_*)\pi}\widetilde{\sigma}_n(x) + \mathcal{O}\left(1/n^2\right), \tag{55}$$

$$v_n(x), z_n(x) = \mathcal{O}\left(1/n^2\right). \tag{56}$$

Proof of Proposition 18: In this proof, we omit μ in subscripts to simplify the notations and we deal with the focusing case (the defocussing case may be treated similarly). From (48) and (45a) we get

$$\begin{aligned}
&u_n(x) = 2\sin[(n+n_*)n\pi x] + \mathcal{O}(1/n^2) \\
&+\tfrac{1}{(n+n_*)\pi}\int_0^x \sin[(n+n_*)\pi(x-s)]\{[(n+n_*)\pi]^2 - \beta_n + \mu - \phi_\mu(s)^2\}2\sin[(n+n_*)\pi s]ds.
\end{aligned}$$

By developing $\sin[(n+n_*)\pi(x-s)]$, we get the conclusion with

$$\begin{aligned}
\rho_n(x) &:= 2\int_0^x \cos[(n+n_*)\pi s]\sin[(n+n_*)\pi s]\{[(n+n_*)\pi]^2 - \beta_n + \mu - \phi_\mu(s)^2\}ds, \\
\sigma_n(x) &:= 2\int_0^x \sin^2[(n+n_*)\pi s]\{[(n+n_*)\pi]^2 - \beta_n + \mu - \phi_\mu(s)^2\}ds,
\end{aligned}$$

that satisfy (52) (see (39)). Note that $\rho_n(1)=0$ as the integral of an odd function. The decomposition of w_n may be proved similarly. From (45a), (49) and (50), we get

$$v_n(x) = -2\int_0^1 G_{\omega_n}(x,\sigma)\phi_\mu^2(\sigma)\sin[(n+n_*)\pi\sigma]d\sigma + \mathcal{O}\left(1/n^2\right).$$

Developing the hyperbolic sinuses, we get

$$\begin{array}{rl} G_{\omega_n}(x,\sigma) = & \frac{1}{2\omega_n}\Big[\Big(-e^{\omega_n(x-\sigma)}+e^{\omega_n(x+\sigma-2)}+e^{-\omega_n(x+\sigma)}\Big)\left(1+\mathcal{O}\left(1/n\right)\right)1_{\sigma>x} \\ & \Big(e^{\omega_n(x+\sigma-2)}+e^{-\omega_n(\sigma+x)}-e^{-\omega_n(x-\sigma)}\Big)\left(1+\mathcal{O}\left(1/n\right)\right)1_{\sigma<x}\Big]. \end{array} \tag{57}$$

In particular, $v_n(x)$ contains terms of the form

$$\begin{array}{rl} & \frac{1}{2\omega_n}\int_x^1 e^{(-\omega_n\pm i(n+n_*)\pi)\sigma}\phi_\mu^2(\sigma)d\sigma e^{\omega_n x} \\ = & -\frac{1}{2\omega_n}\left(\frac{e^{\pm i(n+n_*)\pi x}}{-\omega_n\pm i(n+n_*)\pi}\phi_\mu^2(x)+\int_x^1\frac{e^{\omega_n(x-\sigma)\pm i(n+n_*)\pi\sigma}}{-\omega_n\pm i(n+n_*)\pi}(\phi_\mu^2)'(\sigma)d\sigma\right) \\ = & \mathcal{O}\left(1/n^2\right). \end{array}$$

Working similarly on the other terms of the right hand side of (57), we get $v_n(x)=\mathcal{O}(1/n^2)$. The estimates on w_n and z_n may be proved similarly. □

5.4 Link with $H^3_{(0)}(0,1)$

Proposition 19 *Let $\mu\in(\mp\pi^2,+\infty)$. There exists $C=C(\mu)>0$ such that,*

$$\left(\sum_{n=1}^\infty |n^3\langle Z,\Psi_n^\pm\rangle|^2\right)^{1/2} \leqslant C\|Z\|_{H^3_{(0)}}, \quad \forall Z\in H^3_{(0)}((0,1),\mathbb{C}^2). \tag{58}$$

Proof of Proposition 19: In this proof, we omit μ in subscript to simplify the notations and we deal with the focusing case (the defocussing case may be treated similarly). Let $\mu\in(-\pi^2,+\infty)$.

First step: Existence of $C>0$ such that

$$\left(\sum_{n=1}^\infty |n\langle Z,\Psi_n^\pm\rangle|^2\right)^{1/2} \leqslant C\|Z\|_{H^1_0}, \quad \forall Z\in H^1_0((0,1),\mathbb{C}^2).$$

For $Z\in H^1_0((0,1),\mathbb{C}^2)$, we have $\langle Z,\Psi_n^-\rangle = \langle\widetilde{Z},W_n^+\rangle$ where $\widetilde{Z}:=JZ/2\in H^1_0((0,1),\mathbb{C}^2)$ by Proposition 13 *(viii)*. Using (56), we see that it is sufficient to prove that

$$\left(\sum_{n=1}^\infty \left|n\int_0^1 f(x)w_n(x)dx\right|^2\right)^{1/2} \leqslant C\|f\|_{H^1_0}, \quad \forall f\in H^1_0((0,1),\mathbb{C}).$$

Using integrations by part, (55) and (52), we get

$$\begin{aligned}
\left(\sum_{n=1}^{\infty}\left|n\int_0^1 f(x)w_n(x)dx\right|^2\right)^{1/2} \leqslant\ & \left(\sum_{n=1}^{\infty}\left|n\int_0^1 f(x)\sin[(n+n_*)\pi x]dx\right|^2\right)^{1/2}\\
&+\left(\sum_{n=1}^{\infty}\left|\int_0^1 f(x)\sin[(n+n_*)\pi x]\rho_n(x)dx\right|^2\right)^{1/2}\\
&+\left(\sum_{n=1}^{\infty}\left|\int_0^1 f(x)\cos[(n+n_*)\pi x]\sigma_n(x)dx\right|^2\right)^{1/2}\\
\leqslant\ & C\left(\sum_{n=1}^{\infty}\left|\int_0^1 f'(x)\cos[(n+n_*)\pi x]dx\right|^2\right)^{1/2}\\
&+C\left(\sum_{n=1}^{\infty}\left|\tfrac{1}{n\pi}\int_0^1 (f\rho_n)'(x)(x)\cos[(n+n_*)\pi x]dx\right|^2\right)^{1/2}\\
&+C\left(\sum_{n=1}^{\infty}\left|\tfrac{1}{n\pi}\int_0^1 (f\sigma_n)'(x)\sin[(n+n_*)\pi x]dx\right|^2\right)^{1/2}.
\end{aligned}$$

Bessel-Parseval inequality gives the conclusion.

Second step: Proof of (58). For $Z \in H^3_{(0)}((0,1),\mathbb{C}^2)$ we have $\langle Z, \Psi_n^-\rangle = i\langle \mathcal{L}_\mu Z, \Psi_n^-\rangle/\beta_n$, which gives the conclusion thanks to (39) and the first step. $\square$

5.5 Asymptotic estimates

Proposition 20 *For $\mu \in (\mp\pi^2, +\infty)$ and $n \in \mathbb{N}^*$ we define*

$$\Gamma^+_{n,\mu} := \left\langle (x\phi_\mu)'\begin{pmatrix}1\\0\end{pmatrix}, \Psi^-_{n,\mu}(x)\right\rangle = \int_0^1 (x\phi_\mu)'(x)\overline{\Psi^{(1)}_{n,\mu}(x)}dx. \tag{59}$$

For every $\mu \in (\mp\pi^2, +\infty)$, there exists $C = C(\mu) > 0$ such that

$$\left|\Gamma^+_{n,\mu} - \frac{(-1)^{n+n_*+1}\phi'_\mu(1)}{\pi n}\right| \leqslant \frac{C}{n^2}, \quad \forall n \in \mathbb{N}^* \tag{60}$$

where $n_* = n_*(\mu)$ is as in (39).

Remark 21 *Note that $\phi'_\mu(1) \neq 0$; otherwise, $\phi'_\mu(0)$ would vanish (symmetry of ϕ_μ) and ϕ_μ would be identically zero, because of the uniqueness in Cauchy-Lipschitz theorem. Thus Proposition 20 gives the asymptotic behavior: $\Gamma^+_{n,\mu} \sim (-1)^{n+n_*+1}\phi'_\mu(1)/(\pi n)$ when $n \to +\infty$.*

Proof of Proposition 20: We deduce from (55) that

$$\begin{aligned}
\int_0^1 (x\phi_\mu)'(x)w_n(x)dx = &\int_0^1 (x\phi_\mu)'(x)2\sin[(n+n_*)\pi x]dx\\
&+\int_0^1 (x\phi_\mu)'(x)\Big(\tfrac{\sin[(n+n_*)\pi x]}{(n+n_*)\pi}\widetilde{\rho}_n(x) - \tfrac{\cos[(n+n_*)\pi x]}{(n+n_*)\pi}\widetilde{\sigma}_n(x)\Big)dx + \mathcal{O}\left(\tfrac{1}{n^2}\right).
\end{aligned}$$

Integrating by part each of the 3 terms in the right hand side and using (52) we get

$$\int_0^1 (x\phi_\mu)'(x)w_n(x)dx = \frac{(-1)^{n+n_*+1}2\phi'_\mu(1)}{\pi n} + \mathcal{O}\left(1/n^2\right). \tag{61}$$

Using (56) and Proposition 13*(viii)* we get the conclusion. $\square$

6 Controllability of the linearized system

Proposition 22 *Let $\mu \in (\mp\pi^2, +\infty)$ be such that*

- ***(A)*** *all non zero eigenvalues of $\mathcal{L}_\mu$ are simple,*
- ***(B)*** *$\Gamma^+_{n,\mu} \neq 0, \forall n \in \mathbb{N}^*$ (see (59) for the definition of $\Gamma^+_{n,\mu}$).*

Then the map $d\Theta_{T,\mu}(0) : \dot{H}^1_0((0,T),\mathbb{R}) \to H^3_{(0)}(0,1) \cap (\phi_\mu e^{\pm i\mu T})^\perp$ has a continuous right inverse.

Here, we use the notation

$$(\phi_\mu e^{\pm i\mu T})^\perp := \left\{ \Psi \in L^2(0,1); \mathrm{Re}\left(\int_0^1 \overline{\Psi(x)}\phi_\mu(x)dx e^{\pm i\mu T} \right) = 0 \right\}.$$

Proof of Proposition 22: By Proposition 11, we have

$$d\Theta_{T,\mu}(0).U = \widetilde{\Psi}(T)e^{\pm i\mu T},$$

where $\widetilde{\Psi}$ solves (33). Identifying $H^3_{(0)}((0,1),\mathbb{C})$ with $H^3_{(0)}((0,1),\mathbb{R}^2)$ (by decomposition in real and imaginary parts), we get

$$d\Theta_{T,\mu}(0).U = Z(T)e^{\pm i\mu T},$$

where $Z = (\mathrm{Re}\widetilde{\Psi}, \mathrm{Im}\widetilde{\Psi}) \in C^0([0,T], H^3_{(0)}((0,1),\mathbb{R}^2)) \cap C^1([0,T], H^1_0((0,1),\mathbb{R}^2))$ solves (35).

By Proposition 13 *(i)* and *(vii)*, we have

$$Z(t) = c_0^+(t)\Phi_0^+ + c_0^-(t)\Phi_0^- + \sum_{n\in\mathbb{N}^*} [c_n(t)\Phi_n^+ + \overline{c_n(t)\Phi_n^-}] \quad \text{in } L^2((0,1),\mathbb{C}^2), \quad \forall t \in [0,T],$$

where $c_0^\pm(t) := \langle Z(t), \Psi_0^\pm \rangle$ and $c_n(t) := \langle Z(t), \Psi_n^+ \rangle \in C^1([0,T],\mathbb{C})$ for every $n \in \mathbb{N}$. From the equation (35) we deduce that

$$\begin{aligned}
&\dot{c}_0^-(t) = U(t)\Gamma^-_{0,\mu},\\
&\dot{c}_0^+(t) = c_0^-(t) + U(t)\Gamma^+_{0,\mu},\\
&\dot{c}_n(t) = i\beta_{n,\mu}c_n(t) + U(t)\Gamma^+_{n,\mu}, \quad \forall n \in \mathbb{N}^*,
\end{aligned}$$

where $\Gamma^\pm_{0,\mu} := \int_0^1 (x\phi_\mu)'(x)(\Psi_0^\pm)^{(1)}(x)dx$. Solving these ODEs and using the assumption $\int_0^T U = 0$, we get

$$\begin{aligned}
&c_0^-(T) = 0,\\
&c_0^+(T) = \Gamma^-_{0,\mu}\int_0^T (T-t)U(t)dt,\\
&c_n(T) = e^{i\beta_{n,\mu}T}\Gamma^+_{n,\mu}\int_0^T U(t)e^{-i\beta_{n,\mu}t}dt, \quad \forall n \in \mathbb{N}^*.
\end{aligned}$$

Integrating by parts and using $U(0) = U(T) = 0$ we get

$$\begin{aligned}
&c_0^-(T) = 0,\\
&c_0^+(T) = \Gamma^-_{0,\mu}\int_0^T \tfrac{(T-t)^2}{2}\dot{U}(t)dt,\\
&c_n(T) = e^{i\beta_{n,\mu}T}\tfrac{\Gamma^+_{n,\mu}}{i\beta_{n,\mu}}\int_0^T \dot{U}(t)e^{-i\beta_{n,\mu}t}dt, \quad \forall n \in \mathbb{N}^*.
\end{aligned}$$

By Proposition 31 in Appendix D and (39), there exists a continuous map $L_T : \mathbb{R} \times l^2(\mathbb{N}^*,\mathbb{C}) \to L^2((0,T),\mathbb{R})$ such that, for every $(d_0,(d_n)_{n\in\mathbb{N}^*}) \in \mathbb{R} \times l^2(\mathbb{N}^*,\mathbb{C})$, the function $\nu := L_T(d_0,(d_n))$ solves

$$\left\{\begin{aligned}
&\int_0^T \nu(t)dt = \int_0^T (T-t)\nu(t)dt = 0,\\
&\int_0^T \tfrac{(T-t)^2}{2}\nu(t)dt = d_0,\\
&\int_0^T \nu(t)e^{-i\beta_{n,\mu}t}dt = d_n, \forall n \in \mathbb{N}^*.
\end{aligned}\right.$$

For $\Psi_f \in H^3_{(0)}(0,1)$ such that

$$\mathrm{Re}\left(\int_0^1 \overline{\Psi_f(x)}\phi_\mu(x)e^{\pm i\mu T}dx\right) = 0, \tag{62}$$

we define $d(\Psi_f) := (d_n)_{n\in\mathbb{N}}$ by

$$d_0 := \frac{\langle Z_f, \Psi_0^+\rangle}{\Gamma_{0,\mu}^-}, \qquad d_n := \frac{i\beta_{n,\mu}\langle Z_f, \Psi_n^+\rangle e^{-i\beta_{n,\mu}T}}{\Gamma_{n,\mu}^+}, \quad \forall n \in \mathbb{N}^*$$

where $Z_f := (\mathrm{Re}[\Psi_f e^{\mp i\mu T}], \mathrm{Im}[\Psi_f e^{\mp i\mu T}])$.

We remark that $\Gamma_{0,\mu}^- \neq 0$; indeed the relation (42) and integrations by parts justify that

$$\Gamma_{0,\mu}^- = \int_0^1 (x\phi_\mu)'(x)\phi_\mu(x)dx = \frac{1}{2}\int_0^1 \phi_\mu(x)^2 dx > 0.$$

By assumption ***(B)***, d_n is well defined for every $n \in \mathbb{N}^*$. Using (42) and (62), we see that $d_0 \in \mathbb{R}$. By Proposition 19 and (60), the map $\Psi_f \in H^3_{(0)} \cap (\phi_\mu e^{\pm i\mu T})^\perp \mapsto d(\Psi_f)$ takes values in $l^2(\mathbb{N},\mathbb{C})$. We get the conclusion with $d\Theta_{T,\mu}(0)^{-1}.\Psi_f := L_T[d(\Psi_f)]$. □

7 Genericity

In this section we verify that the assumptions ***(A)*** and ***(B)*** in Proposition 22 hold generically with respect to the parameter μ.

Proposition 23 *There exists a countable set $J \subset (\mp\pi^2, +\infty)$ such that, for every $\mu \in (\mp\pi^2, +\infty) \setminus J$, all non zero eigenvalues of $\mathcal{L}_\mu$ are simple, and $\Gamma_{n,\mu}^+ \neq 0, \forall n \in \mathbb{N}^*$.*

7.1 Reformulation of the problem

The purpose of the next two statements is to recast conditions ***(A)*** and ***(B)*** such that they become amenable to complex-variable methods. This is accomplished in (66) below.

Proposition 24 *Let $\mu \in (\mp\pi^2, +\infty)$. We denote $\Psi_n^\pm = \binom{f_{n,\mu}(x)}{\mp i g_{n,\mu}(x)}$. Then*

$$\Gamma_{n,\mu}^+ = \frac{\phi_\mu'(1)g_{n,\mu}'(1)}{\beta_{n,\mu}}, \quad \forall n \in \mathbb{N}^*.$$

Proof of Proposition 24: In this proof, we omit μ in subscript to simplify the notations, and we treat the focusing case (the defocussing one may be treated similarly). From the relation $\mathcal{L}_\mu \Psi_n^\pm = \mp i\beta_n \Psi_n^\pm$, we get

$$\begin{array}{ll} f_n'' + (\phi_\mu^2 - \mu)f_n = \beta_{n,\mu}g_n, & f_n(0) = f_n(1) = 0, \\ g_n'' + (3\phi_\mu^2 - \mu)g_n = \beta_{n,\mu}f_n, & g_n(0) = g_n(1) = 0. \end{array} \tag{63}$$

So, integration by parts gives

$$\begin{aligned} \Gamma_n^+ &= \int_0^1 (x\phi_\mu)(x)f_n(x)dx \\ &= \frac{1}{\beta_n}\int_0^1 \Big([\partial_x^2 + 3\phi_\mu^2 - \mu]g_n\Big)(x)(x\phi_\mu)'(x)dx \\ &= \frac{\phi_\mu'(1)g_n'(1)}{\beta_n} + \frac{1}{\beta_n}\int_0^1 \Big([\partial_x^2 + 3\phi_\mu^2 - \mu](x\phi_\mu)'\Big)(x)g_n(x)dx. \end{aligned}$$

Moreover, using (13), we get

$$[\partial_x^2 + 3\phi_\mu^2 - \mu](x\phi_\mu)' = x(\phi_\mu'' + \phi_\mu^3 - \mu\phi_\mu)' + 3(\phi_\mu'' + \phi_\mu^3 - \mu\phi_\mu) + 2\mu\phi_\mu = 2\mu\phi_\mu$$

and

$$\int_0^1 \phi_\mu(x) g_n(x)dx \;=\; \frac{1}{\beta_n}\int_0^1 [\partial_x^2 + \phi_\mu^2 - \mu] f_n \phi_\mu dx = \frac{1}{\beta_n}\int_0^1 [\partial_x^2 + \phi_\mu^2 - \mu]\phi_\mu f_n dx = 0,$$

which gives the conclusion. □

Proposition 25 *Let $\mu \in (\mp\pi^2, +\infty)$, $n \in \mathbb{N}^*$, $(f_{n,\mu}^{[1]}, g_{n,\mu}^{[1]})$, $(f_{n,\mu}^{[2]}, g_{n,\mu}^{[2]})$ be the solutions of*

$$\begin{cases} f'' \pm (\phi_\mu^2 - \mu) f = \beta_{n,\mu} g, \\ g'' \pm (3\phi_\mu^2 - \mu) g = \beta_{n,\mu} f, \end{cases} \tag{64}$$

associated to the following initial conditions at $x = 0$,

$$\begin{array}{ll} f_{n,\mu}^{[1]}(0) = g_{n,\mu}^{[1]}(0) = (g_{n,\mu}^{[1]})'(0) = 0, & (f_{n,\mu}^{[1]})'(0) = 1, \\ f_{n,\mu}^{[2]}(0) = g_{n,\mu}^{[2]}(0) = (f_{n,\mu}^{[2]})'(0) = 0, & (g_{n,\mu}^{[2]})'(0) = 1, \end{array} \tag{65}$$

and

$$A_{n,\mu} := \begin{pmatrix} f_{n,\mu}^{[1]}(1) & f_{n,\mu}^{[2]}(1) \\ g_{n,\mu}^{[1]}(1) & g_{n,\mu}^{[2]}(1) \end{pmatrix}.$$

(i) If $A_{n,\mu}$ is not the zero matrix, then $i\beta_{n,\mu}$ is a simple eigenvalue of $\mathcal{L}_\mu$.

(ii) If the first column of $A_{n,\mu}$ is not the zero vector $\binom{0}{0}$ then $\Gamma_{n,\mu}^+ \neq 0$.

In particular,

$$f_{n,\mu}^{[1]}(1) \neq 0 \quad \Rightarrow \quad \beta_{n,\mu} \text{ is simple and } \Gamma_{n,\mu}^+ \neq 0. \tag{66}$$

Proof of Proposition 25: Let $(f_{n,\mu}^{[3]}, g_{n,\mu}^{[3]})$, $(f_{n,\mu}^{[4]}, g_{n,\mu}^{[4]})$ be the solutions of (64) such that

$$\begin{array}{ll} g_{n,\mu}^{[3]}(0) = (f_{n,\mu}^{[3]})'(0) = (g_{n,\mu}^{[3]})'(0) = 0, & f_{n,\mu}^{[3]}(0) = 1, \\ f_{n,\mu}^{[4]}(0) = (f_{n,\mu}^{[4]})'(0) = (g_{n,\mu}^{[4]})'(0) = 0, & g_{n,\mu}^{[4]}(0) = 1. \end{array}$$

We assume that $i\beta_{n,\mu}$ is not a simple eigenvalue of $\mathcal{L}_\mu^*$. Then (see Proposition 13 *(v)*) there exists two linearly independent solutions $(f_{n,\mu}, g_{n,\mu})$ and $(\tilde{f}_{n,\mu}, \tilde{g}_{n,\mu})$ of (63). They may be expanded with respect to the fundamental system

$$\begin{pmatrix} f_{n,\mu} \\ g_{n,\mu} \end{pmatrix} = \sum_{k=1}^4 a_k \begin{pmatrix} f_{n,\mu}^{[k]} \\ g_{n,\mu}^{[k]} \end{pmatrix} \text{ and } \begin{pmatrix} \tilde{f}_{n,\mu} \\ \tilde{g}_{n,\mu} \end{pmatrix} = \sum_{k=1}^4 \tilde{a}_k \begin{pmatrix} f_{n,\mu}^{[k]} \\ g_{n,\mu}^{[k]} \end{pmatrix} \text{ with } a_k, \tilde{a}_k \in \mathbb{C} \text{ for } k = 1, ..., 4.$$

From the property $f_{n,\mu}(0) = g_{n,\mu}(0) = \tilde{f}_{n,\mu}(0) = \tilde{g}_{n,\mu}(0) = 0$ we deduce that $a_3 = a_4 = \tilde{a}_3 = \tilde{a}_4 = 0$. From the property $f_{n,\mu}(1) = g_{n,\mu}(1) = \tilde{f}_{n,\mu}(1) = \tilde{g}_{n,\mu}(1) = 0$, we deduce that the two linearly independent vectors $\binom{a_3}{a_4}$ and $\binom{\tilde{a}_3}{\tilde{a}_4}$ belong to the kernel of $A_{n,\mu}$. Thus $A_{n,\mu} = 0$. This proves the first statement.

We assume that $A_{n,\mu} \neq 0$ and $\Gamma_{n,\mu}^+ = 0$. By Proposition 24, we have $g_{n,\mu}'(1) = 0$. Since the eigenvalue $i\beta_{n,\mu}$ is simple, Ψ_n^+ is either odd or even. In particular $g_{n,\mu}'(0) = \pm g_{n,\mu}'(1) = 0$. Thus, Ψ_n is collinear to $\binom{f_{n,\mu}^{[1]}}{g_{n,\mu}^{[1]}}$. We deduce from the relation $\Psi_n^+(1) = 0$ that the first column of $A_{n,\mu}$ vanishes. This proves the second statement. □

7.2 Analyticity of the eigenvalues

In this section we state the analytic dependence of the eigenvalues of $\mathcal{L}_\mu$ with respect to μ. Because of the non–selfadjointness of $\mathcal{L}_\mu$ and $\mathcal{M}_\mu$, this property is not at all obvious. In fact, there are simple examples of analytic families of 2×2 matrices whose eigenvalues are not analytic functions of the parameter (see e.g. [29]). We therefore provide a proof in Appendix C.

Proposition 26 *There exists continuous functions $F_n : [\mp\pi^2, \infty) \to \mathbb{R}_+^*$, for $n \in \mathbb{N}^*$, that are analytic on $(\mp\pi^2, \infty)$ such that*

- $\{F_n(\mu); n \in \mathbb{N}^*\} = \{\beta_{n,\mu}; n \in \mathbb{N}^*\}$, *for every $\mu \in (\mp\pi^2, \infty)$,*
- $F_n(\mp\pi^2) = [(n+1)^2 - 1]\pi^2$.

7.3 Proof of Proposition 23

The proof of Proposition 23 follows from (66) and the next result.

Proposition 27 *There exists a countable set $J \subset (\mp\pi^2, +\infty)$ such that, for every $\mu \in (\mp\pi^2, +\infty) \setminus J$ and $n \in \mathbb{N}^*$, the solution $(f_{n,\mu}^{[1]}, g_{n,\mu}^{[1]})$ of (64)-(65) satisfies $f_{n,\mu}^{[1]}(1) \neq 0$.*

Proof of Proposition 27: We treat the focusing case (the defocussing one may be treated similarly). Let $(F_n)_{n\in\mathbb{N}^*}$ be as in Proposition 26. We denote by $(k_{n,\mu}, h_{n,\mu})$ the solution of

$$\left\{\begin{array}{l} k_{n,\mu}'' + (\phi_\mu^2 - \mu)k_{n,\mu} = F_n(\mu)h_{n,\mu}, \\ h_{n,\mu}'' + (3\phi_\mu^2 - \mu)h_{n,\mu} = F_n(\mu)k_{n,\mu}, \\ k_{n,\mu}(0) = h_{n,\mu}(0) = h_{n,\mu}'(0) = 0, \quad k_{n,\mu}'(0) = 1, \end{array}\right. \tag{67}$$

and we introduce the map

$$\left| \begin{array}{cccc} G_n : & [-\pi^2, +\infty) & \to & \mathbb{R} \\ & \mu & \mapsto & k_{n,\mu}(1). \end{array}\right.$$

First step: G_n is analytic for every $n \in \mathbb{N}^*$. Let $n \in \mathbb{N}^*$. Since n is fixed in all this step, we will write k_μ, h_μ, F instead of $k_{n,\mu}$, $h_{n,\mu}$, F_n. Let $\mu_0 \in (-\pi^2, +\infty)$. The functions $\mu \mapsto \phi_\mu$ and $\mu \mapsto F(\mu)$ may be extended as holomorphic functions of $\mu \in \Omega$ where $\Omega := \{\mu \in \mathbb{C}; \mu_0 - \epsilon < \mathrm{Re}(\mu) < \mu_0 + \epsilon, -\epsilon < \mathrm{Im}(\mu) < \epsilon\}$ for some $\epsilon > 0$, by the sum of the converging Taylor series at μ_0. For $\mu \in \Omega$, we introduce the notations

$$\begin{gathered} \mu_1 := \mathrm{Re}(\mu), \quad \mu_2 := \mathrm{Im}(\mu), \quad F^{(1)}(\mu) := \mathrm{Re}[F(\mu)], \quad F^{(2)}(\mu) := \mathrm{Im}[F(\mu)], \\ a_1(x) := \mathrm{Re}[\phi_\mu(x)^2 - \mu], \quad a_2(x) := \mathrm{Im}[\phi_\mu(x)^2 - \mu], \\ b_1(x) := \mathrm{Re}[3\phi_\mu(x)^2 - \mu], \quad b_2(x) := \mathrm{Im}[3\phi_\mu(x)^2 - \mu], \\ k_\mu^{(1)}(x) := \mathrm{Re}[k_\mu(x)], \quad k_\mu^{(2)}(x) := \mathrm{Im}[k_\mu(x)], \quad h_\mu^{(1)}(x) := \mathrm{Re}[h_\mu(x)], \quad h_\mu^{(2)}(x) := \mathrm{Im}[h_\mu(x)]. \end{gathered}$$

We deduce from (67) that, for every $\mu \in \Omega$,

$$(k_\mu^{(1)})'' + a_1 k_\mu^{(1)} - a_2 k_\mu^{(2)} = F^{(1)} h_\mu^{(1)} - F^{(2)} h_\mu^{(2)}, \tag{68a}$$

$$(k_\mu^{(2)})'' + a_1 k_\mu^{(2)} + a_2 k_\mu^{(1)} = F^{(1)} h_\mu^{(2)} + F^{(2)} h_\mu^{(1)}, \tag{68b}$$

$$(h_\mu^{(1)})'' + b_1 h_\mu^{(1)} - b_2 h_\mu^{(2)} = F^{(1)} k_\mu^{(1)} - F^{(2)} k_\mu^{(2)}, \tag{68c}$$

$$(h_\mu^{(2)})'' + b_1 h_\mu^{(2)} + b_2 h_\mu^{(1)} = F^{(1)} k_\mu^{(2)} + F^{(2)} k_\mu^{(1)}. \tag{68d}$$

(68e)

In particular, for every $(\mu_1, \mu_2) \in \widetilde{\Omega} := (\mu_0 - \epsilon, \mu_0 + \epsilon) \times (-\epsilon, \epsilon)$, the function $Y_\mu := (k_\mu^{(1)}, k_\mu^{(2)}, h_\mu^{(1)}, h_\mu^{(2)})$ solves an equation of the form

$$\left\{\begin{array}{l} \frac{d^2 Y_\mu}{dx^2} = \mathcal{F}(x, Y_\mu, \mu_1, \mu_2) \\ Y_\mu(0) = (0,0,0,0), \\ Y_\mu'(0) = (1,0,0,0). \end{array}\right. \tag{69}$$

where the function $\mathcal{F}$ is of class C^1 with respect to $(x, Y, \mu_1, \mu_2) \in (0,1) \times \mathbb{R}^4 \times \widetilde{\Omega}$, thus Y_μ has partial derivatives with respect to μ_1 and μ_2. Now, we prove that they satisfy the Cauchy-Riemann relations, in order to get the holomorphy of $\mu \in \Omega \mapsto Y_\mu(1)$. We introduce the functions

$$K_{i,j} := \frac{\partial k_\mu^{(i)}}{\partial \mu_j}, \quad H_{i,j} := \frac{\partial h_\mu^{(i)}}{\partial \mu_j}, \quad \forall i, j \in \{1,2\}.$$

Computing $\partial_{\mu_1}(68a) - \partial_{\mu_2}(68b)$, $\partial_{\mu_2}(68a) + \partial_{\mu_1}(68b)$, $\partial_{\mu_1}(68c) - \partial_{\mu_2}(68d)$, $\partial_{\mu_2}(68c) + \partial_{\mu_1}(68d)$, and using the Cauchy-Riemann relations on a_1, a_2, b_1, b_2, $F^{(1)}$, $F^{(2)}$, we get

$$\begin{array}{l}
(K_{1,1}-K_{2,2})'' + a_1(K_{1,1}-K_{2,2}) - a_2(K_{2,1}+K_{1,2}) = F^{(1)}(H_{1,1}-H_{2,2}) - F^{(2)}(H_{1,2}+H_{2,1}),\\
(K_{1,2}+K_{2,1})'' + a_1(K_{1,2}+K_{2,1}) - a_2(K_{2,2}-K_{1,1}) = F^{(1)}(H_{1,2}+H_{2,1}) - F^{(2)}(H_{2,2}-H_{1,1}),\\
(H_{1,1}-H_{2,2})'' + b_1(H_{1,1}-H_{2,2}) - b_2(H_{2,1}+H_{1,2}) = F^{(1)}(K_{1,1}-K_{2,2}) - F^{(2)}(K_{1,2}+K_{2,1}),\\
(H_{1,2}+K_{2,1})'' + b_1(H_{1,2}+H_{2,1}) - b_2(H_{2,2}-H_{1,1}) = F^{(1)}(K_{1,2}+K_{2,1}) - F^{(2)}(K_{2,2}-K_{1,1}),\\
(K_{1,1}-K_{2,2}, K_{1,2}+K_{2,1}, H_{1,1}-H_{2,2}, H_{1,2}+H_{2,1})(0) = (0,0,0,0),\\
(K_{1,1}-K_{2,2}, K_{1,2}+K_{2,1}, H_{1,1}-H_{2,2}, H_{1,2}+H_{2,1})'(0) = (0,0,0,0).
\end{array}$$

The uniqueness of the solution of this linear system ensures that $K_{1,1}-K_{2,2} = K_{1,2}+K_{2,1} = H_{1,1}-H_{2,2} = H_{1,2}+H_{2,1} = 0$. In particular $(K_{1,1}-K_{2,2})(1) = (K_{1,2}+K_{2,1})(1)$, which proves the holomorphy of G_n on Ω.

Second step: $G_n(-\pi^2) \neq 0$, $\forall n \in \mathbb{N}^*$. Let $n \in \mathbb{N}^*$. Thanks to (17) and the second conclusion of Proposition 26, we have $G_{n+1}(-\pi^2) = f(1)$ where (f,g) is the solution of the Cauchy problem

$$\left\{\begin{array}{l}
f'' + \pi^2 f = (n^2-1)\pi^2 g,\\
g'' + \pi^2 g = (n^2-1)\pi^2 f,\\
f(0) = g(0) = g'(0) = 0, \quad f'(0) = 1.
\end{array}\right. \tag{70}$$

This system may be written

$$\left\{\begin{array}{l}
\left(\frac{d^2}{dx^2} + \pi^2\right)^2 f = (n^2-1)^2\pi^4 f,\\
f(0) = f''(0) = 0, \quad f'(0) = 1, \quad f^{(3)}(0) = -\pi^2.
\end{array}\right.$$

Thus, f may be computed explicitly. In particular,

$$f(1) = \left\{\begin{array}{l}
-(1 + 2/(\pi-3)) \quad \text{if } n = 1,\\
3/4 \quad \text{if } n = 2,\\
\sinh(\sqrt{n^2-2}\pi)/(2\pi\sqrt{n^2-2}) \quad \text{if } n \geqslant 3,
\end{array}\right.$$

which gives the conclusion.

Third step: Conclusion. By the isolated zero principle, for every $n \in \mathbb{N}^*$, there exists a countable set $J_n \subset (-\pi^2, +\infty)$ such that, $G_n(\mu) \neq 0$ for every $\mu \in (-\pi^2, +\infty) \setminus J_n$. Then, $J := \cup_{n \in \mathbb{N}^*} J_n$ gives the conclusion. □

8 Proof of the main result

Let J be as in Proposition 23, $\mu \in (\mp\pi^2, +\infty) \setminus J$ and $T > 0$. By Proposition 11, the map

$$\Theta_{T,\mu} : \dot{H}_0^1((0,T), \mathbb{R}) \to H^3_{(0)}(0,1) \cap \mathcal{S}_{\|\phi_\mu\|_{L^2}}$$

is C^1. By Proposition 22 and (23),

$$d\Theta_{T,\mu}(0) : \dot{H}_0^1((0,T), \mathbb{R}) \to H^3_{(0)}(0,1) \cap (\phi_\mu e^{\pm i\mu T})^\perp$$

has a continuous right inverse. The inverse mapping theorem gives the conclusion.

9 Conclusion and perspectives

Motivated by the control of Bose–Einstein condensates, we have studied the controllability of the nonlinear Schrödinger equation (focusing and defocusing) with a bilinear control term arising from manipulating the size of a "hard-wall" (box) trap. We showed that local exact controllability around the ground state holds generically with respect to the parameter μ. Since μ is a parameter associated with the *transformed* problem (7), this leaves the question of whether genericity also holds with respect to the system parameter κ of the *original* problem (1). This is indeed so, as is readily seen from the identity

$$\|\phi_\mu\|^2_{L^2(0,1)} = \frac{2\kappa m}{\hbar^2}$$

and the convexity condition (16)[3]. While the genericity property implies that local controllability holds with "probability one w.r.t. *random* choices" of μ (or κ), for any *particular* value of μ (resp. κ) Theorem 2 cannot be applied directly. It will be shown elsewhere [20] how rigorous numerical computation can be utilized in these cases.

Of the numerous possible generalizations of the control problem considered in the present paper we briefly mention three:

(i) more *general nonlinearities*;

(ii) controllability around *excited* states[4];

(iii) *global* exact controllability.

In (i) and (ii) several steps of our approach will need to be adapted, such as the study of the spectrum of the operator $\mathcal{L}_\mu$, which may no longer be purely imaginary, or the proof of the genericity result in Section 7, which uses the convexity inequality (16). We conjecture that (i) can be handled for "benign" cases such as certain power nonlinearities and that (ii) holds at least in the defocusing case. To prove (iii) one may try to adapt the techniques of [41], although, due to the nonlinearity of the equation, significant new ideas will be required.

A Ground states: proof

In this section we prove Proposition 4.

First, we treat the focusing case. Let $\mu \in (-\pi^2, +\infty)$. There exists a unique solution $w_\mu \in M$ of the minimization problem

$$\begin{array}{r}J_\mu(w_\mu) = \inf\left\{J_\mu(\varphi); \varphi \in M\right\}, \\ J_\mu(\varphi) := \int_0^1 [\varphi'(x)^2 + \mu\varphi(x)^2]dx, \\ M := \left\{\varphi \in H^1_0((0,1),\mathbb{R}); \int_0^1 \varphi(x)^4 dx = 1\right\},\end{array} \tag{71}$$

and a Lagrange multiplier $\alpha_\mu \in \mathbb{R}$ such that

$$\left\{\begin{array}{l} -w_\mu'' + \mu w_\mu = \alpha_\mu w_\mu^3, \\ w_\mu(0) = w_\mu(1) = 0. \end{array}\right. \tag{72}$$

Then

$$\alpha_\mu \int_0^1 w_\mu(x)^4 dx = \int_0^1 \left(w_\mu'(x)^2 + \mu w_\mu(x)^2\right)dx > 0;$$

[3] Another possible parameter is the initial and final size of the trap, which in (3) was set to one for convenience.

[4] These are (real-valued) solutions of (13), with a positive number of zeros ("nodes") within the interval $(0,1)$ [18, 19]. (The node-less solution is the ground state.)

thus $\alpha_\mu > 0$ and $\phi_\mu := \sqrt{\alpha_\mu}\varphi_\mu$ gives the solution. An explicit formula of ϕ_μ is available in terms of Jacobian elliptic functions. For $\mu \in (-\pi^2, +\infty)$, we first find the solution $k = k(\mu)$ of the equation

$$\mu = 4(2k^2 - 1)K(k)^2$$

where K denotes the complete elliptic integral of the first kind (see e.g. [1]). Note that the function $K : [0,1), \to [\pi/2, +\infty)$ is continuous, analytic on $(0,1)$, bijective and $K' > 0$ on $(0,1)$. Thus, the reciprocal $k = k(\mu)$ defines a function $k : [-\pi^2, +\infty) \to [0,1)$ continuous, analytic on $(-\pi^2, +\infty)$, bijective with $k' > 0$ on $(0, +\infty)$. Then, the function ϕ_μ is given by the formula [19]

$$\phi_\mu(x) = 2\sqrt{2}kK(k)\,\mathrm{cn}\left(2K(k)\Big(x - \frac{1}{2}\Big), k\right),$$

where cn is the elliptic cosine function. This proves the analyticity of the map $\mu \in (-\pi^2, +\infty) \mapsto \phi_\mu \in L^2(0,1)$ and the relation

$$\int_0^1 \phi_\mu(x)^2 dx = 8k^2 K(k) \int_0^{K(k)} \mathrm{cn}(y)^2 dy = 8K(k)F(k)$$

where $F(k) := E(k) - (1-k^2)K(k)$ and E is the complete elliptic integral of the second kind. The function F is positive and satisfies $F'(k) = kK(k)$ on $(0,1)$, thus

$$2\langle \partial_\mu \phi_\mu, \phi_\mu \rangle = 8k'(\mu)\Big(K'[k(\mu)]F[k(\mu)] + K[k(\mu)]F'[k(\mu)]\Big) > 0, \quad \forall \mu \in (-\pi^2, +\infty).$$

Note that, when μ tends to $-\pi^2$, then $k(\mu) \to 0$, and $K[k(\mu)]$ is bounded, which proves (17).

In the defocussing case we will not need the variational description of the ground state, so we omit this point. Again, an explicit formula of ϕ_μ is available in terms of Jacobian elliptic functions. For $\mu \in (\pi^2, +\infty)$, we first find the solution $k = k(\mu)$ of the equation

$$\mu = 4(k^2 + 1)K(k)^2.$$

This defines a function $k : [\pi^2, +\infty) \to [0,1)$ continuous, analytic on $(\pi^2, +\infty)$, such that $k' > 0$ on $(\pi^2, +\infty)$. Then [18]

$$\phi_\mu(x) := 2\sqrt{2}kK(k)\,\mathrm{sn}\Big(2K(k)x, k\Big),$$

where sn is the Jacobian elliptic sine function. This proves the analyticity of $\mu \in (\pi^2, +\infty) \mapsto \phi_\mu \in L^2(0,1)$ and the relation

$$\int_0^1 \phi_\mu(x)^2 dx = 4k^2 K(k) \int_0^{K(k)} \mathrm{sn}(y,k)^2 dy = 8K(k)G(k)$$

where $G(k) := E(k) - K(k)$ is positive and satisfies $G'(k) = kE(k)/(1-k^2)$ for every $k \in (0,1)$. The proof may be ended as above. □

B Basic spectral properties: proof

In this appendix we provide the proof of Proposition 13. Our proof is similar to the one for the whole space case, which has been studied extensively; its elements are taken from [32], [36, Appendix B] and adapted from [42].

B.1 Preliminaries

Proposition 28 *In both focusing and defocussing cases, we have*

$$Ker(L_\mu^-) = \mathbb{C}\phi_\mu, \qquad Ker(L_\mu^+) = \{0\}, \quad \forall \mu \in (\mp\pi^2, \infty). \tag{73}$$

In the focusing case,

$$L_\mu^+ \text{ has only one negative eigenvalue, } \forall \mu \in (-\pi^2, \infty). \tag{74}$$

In the defocussing case,

$$L_\mu^+ > 0, \quad \forall \mu \in (\pi^2, +\infty). \tag{75}$$

Proof of Proposition 28:
First step: Proof of $Ker(L_\mu^-) = \mathbb{C}\phi_\mu$. We recall that $\text{Ker}[L_\mu^-] := \{w \in H^2 \cap H_0^1(0,1); (\partial_x^2 \pm \phi_\mu^2 \mp \mu)w \equiv 0\}$. The linear map

$$\left| \begin{array}{ccc} \text{Ker}[L_\mu^-] & \to & \mathbb{C} \\ w & \mapsto & w'(0) \end{array} \right.$$

is injective, thanks to the uniqueness in Cauchy-Lipschitz theorem. Thus $\dim[\text{Ker}(L_\mu^-)] \leqslant 1$. Clearly, $L_\mu^- \phi_\mu = 0$, which gives the conclusion.

Second step: Proof of $Ker(L_\mu^+) = \{0\}$ and (74) in the focusing case. This will be achieved thanks to Step 2.1, Step 2.2 and Step 2.3 below.

Step 2.1: L_μ^+ has at least one negative eigenvalue. This follows from

$$\langle L_\mu^+ \phi_\mu, \phi_\mu \rangle = -2\|\phi_\mu\|_{L^4}^4 < 0$$

and the minimax principle.

Step 2.2: $\langle L_\mu^+ \eta, \eta \rangle \geqslant 0, \forall \eta \perp \phi_\mu^3$. We use the characterization of ϕ_μ by the minimization problem (71). Let $\eta \in H^2 \cap H_0^1((0,1),\mathbb{R})$ be such that $\eta \perp \phi_\mu^3$ in $L^2(0,1)$. Let $z \mapsto w(.,z) \in H^2 \cap H_0^1((0,1),\mathbb{R})$ be a smooth curve such that $w(.,0) = w_\mu$, $\dot{w} := \partial_z[w(.,z)]_{z=0} = \eta$, $\|w(.,z)\|_{L^4(0,1)} \equiv 1$. Since w_μ solves the minimization (71), the function $z \mapsto J[w(.,z)]$ has its minimum at $z = 0$; thus

$$\begin{aligned} 0 &= \frac{d}{dz}\Big[J_\mu[w(.,z)]\Big]_{z=0} = \int_0^1 \Big(w_x \dot{w}_x + \mu w \dot{w}\Big)dx, \\ 0 &\leqslant \frac{d^2}{dz^2}\Big[J_\mu[w(.,z)]\Big]_{z=0} = \int_0^1 \Big(\dot{w}_x^2 + w_x \ddot{w}_x + \mu \dot{w}^2 + \mu w \ddot{w}\Big)dx. \end{aligned} \tag{76}$$

Moreover, $w(.,z) \in M$ for every z, thus

$$0 = \int_0^1 w^3 \dot{w} dx = \int_0^1 \Big(3w^2\dot{w}^2 + w^2\ddot{w}\Big)dx.$$

The Euler-Lagrange equation (72) and the previous relation give, at $z = 0$,

$$\int_0^1 (-w_{xx} + \mu w)\ddot{w} dx = \alpha_\mu \int_0^1 w^3 \ddot{w} dx = -3\alpha_\mu \int_0^1 w^2 \dot{w}^2 = -3\int_0^1 \phi_\mu^2 \eta^2.$$

Incorporating this relation in (76) gives

$$\begin{aligned} 0 &\leqslant \int_0^1 \Big((-\dot{w}_{xx} + \mu\dot{w})\dot{w} + (-w_{xx} + \mu w)\ddot{w}\Big)dx \\ &= \int_0^1 \Big(-\eta_{xx} + \mu\eta - 3\phi_\mu^2\eta\Big)\eta dx \;=\; \langle L_\mu^+ \eta, \eta \rangle. \end{aligned}$$

Step 2.3: The second eigenvalue λ_2 of L_μ^+ is > 0. Thanks to Step 2.2, we know that the second eigenvalue of L_μ^+ is $\geqslant 0$. Let us assume that $\lambda_2 = 0$. Let $v \in D(L_\mu^+)$ be such that $L_\mu^+ v = 0$. By symmetry of ϕ_μ with respect to $x = 1/2$, one may assume that v is odd or even (with respect to $x = 1/2$). If v is odd, then $v(1/2) = 0$ and $v = c\phi_\mu$ for some $c \in \mathbb{R}$, thanks to ODE solutions uniqueness. But this is impossible because ϕ_μ' does not vanish at $x = 0$ and $x = 1$. Thus v is even. The function v has one zero in $(0,1)$ (second eigenvalue of a Sturm-Liouville operator). By symmetry, this must occur at $x = 1/2$, which is impossible as saw before.

Third step: Proof of (75) in the defocussing case. We prove by contradiction that the smallest eigenvalue is positive. To this end, let E be the smallest eigenvalue, $u \in H^2 \cap H_0^1((0,1),\mathbb{R}) \setminus \{0\}$ a corresponding eigenfunction, and assume $E \leq 0$. Then u may be assumed to be positive on $(0,1)$ because it is the ground state of L_μ^+. Thus $\langle u, \phi_\mu \rangle > 0$ and so

$$0 \geq E\langle u, \phi_\mu \rangle = \langle L_\mu^+ u, \phi_\mu \rangle = \langle u, L_\mu^+ \phi_\mu \rangle = 2\int_0^1 u(x)\phi_\mu(x)^3 dx > 0,$$

which is impossible. Therefore $L_\mu^+ > 0$. □

B.2 Statements *(i)* and *(iii)*

The operator $\mathcal{D}$ defined by (38) is self-adjoint, with compact resolvent and simple eigenvalues with an infinite asymptotic gap:

$$\mathcal{D}\begin{pmatrix} \sin(n\pi x) \\ 0 \end{pmatrix} = (n\pi)^2 \begin{pmatrix} \sin(n\pi x) \\ 0 \end{pmatrix}, \qquad \mathcal{D}\begin{pmatrix} 0 \\ \sin(n\pi x) \end{pmatrix} = -(n\pi)^2 \begin{pmatrix} 0 \\ \sin(n\pi x) \end{pmatrix}, \quad \forall n \in \mathbb{N}^*.$$

The operator $\widetilde{\mathcal{M}}_\mu$ is bounded on $L^2((0,1),\mathbb{C}^2)$. By applying [32, Chapter V, paragraph 3, Theorem 4.15.a on Page 293]), we get the first statement of Proposition 13 and the third one, assuming that the second one holds (which will be proved independently below).

B.3 Statement *(ii)*

This proof follows the one of [42, Lemma 12.11], in the case of NLS on the whole line. In order to simplify the notations, we do not write μ in subscript. Let us consider the operator

$$\mathcal{L}^2 = -\begin{pmatrix} \mathcal{T}^* & 0 \\ 0 & \mathcal{T} \end{pmatrix} \quad \text{where} \quad D(\mathcal{T}) := H^4_{(0)}(0,1), \qquad \mathcal{T} := L^+L^-. \tag{77}$$

First step: $Sp(\mathcal{T}) \subset \mathbb{R}$. Let $E \in \mathbb{C} - \{0\}$ be an eigenvalue of $\mathcal{T}$ and ψ be an associated eigenvector:

$$\mathcal{T}\psi = E\psi. \tag{78}$$

Then, $\psi = \psi_1 + c\phi_\mu$, where $\psi_1 \perp \phi_\mu$ and $\psi_1 \neq 0$ (because $L^-\phi_\mu = 0$). Thus, (78) gives

$$\Big[(L^-)^{1/2}L^+(L^-)^{1/2}\Big]\Big((L^-)^{1/2}\psi_1\Big) = E\Big((L^-)^{1/2}\psi_1\Big). \tag{79}$$

Moreover $(L^-)^{1/2}\psi_1 \neq 0$ thanks to (73). Thus E is an eigenvalue of the symmetric operator $(L^-)^{1/2}L^+(L^-)^{1/2}$, so $E \in \mathbb{R}$.

Second step: (74) implies $Sp(\mathcal{T}) \subset \mathbb{R}^+$ in the focusing case. The map

$$g(E) := \langle (L^+ - E)^{-1}\phi_\mu, \phi_\mu \rangle$$

is well defined for $E \in (-E^*, 0]$, where $-E^*$ is the negative eigenvalue of L^+. Moreover, we have

$$g'(E) = \|(L^+ - E)^{-1}\phi_\mu\|^2 \geqslant 0, \quad \forall E \in (-E^*, 0),$$

$$g(0) = -\langle \partial_\mu \phi_\mu, \phi_\mu \rangle < 0,$$

thanks to (16), thus

$$g(E) < 0, \quad \forall E \in (-E^*, 0). \tag{80}$$

We prove by contradiction that the eigenvalues of $\mathcal{T}$ are $\geqslant 0$. We assume that $\mathcal{T}$ has a negative eigenvalue $E < 0$. From (79), we deduce that $(L^-)^{1/2} L^+ (L^-)^{1/2}$ has a negative eigenvalue in $\mathrm{Ker}[L^-]^\perp$; there exists $\zeta \in \mathrm{Ker}[L^-]^\perp$ such that

$$\langle (L^-)^{1/2} L^+ (L^-)^{1/2} \zeta, \zeta \rangle = \langle L^+ \xi, \xi \rangle < 0$$

with $\xi := (L^-)^{1/2}\zeta$. Let P^- be the orthogonal projection from L^2 to $\mathrm{Ker}(L^-)^\perp$. Thanks to the Rayleigh principle, the operator $P^- L^+ P^-$ has a negative eigenvalue $E_3 \in [-E^*, 0)$: $L^+\psi = E_3 \psi + c\phi_\mu$ for some $\psi \perp \phi_\mu$, $\psi \neq 0$, $c \in \mathbb{C}$. If $c = 0$, then ψ is the ground state of L^+ thus $\psi > 0$; in particular the two positive functions ψ and ϕ_μ cannot be orthogonal in $L^2(0,1)$: contradiction. Thus, $c \neq 0$ and $(L^+ - E_3)^{-1}\phi_\mu = \frac{\psi}{c}$. In particular, we have

$$g(E_3) = \langle (L^+ - E_3)^{-1} \phi_\mu, \phi_\mu \rangle = \frac{1}{c} \langle \psi, \phi_\mu \rangle = 0.$$

which is impossible in view of (80). Therefore, the eigenvalues of $\mathcal{T}$ are $\geqslant 0$.

Third step: $Sp(\mathcal{T}) \subset \mathbb{R}^+$ in the defocussing case. Let us assume that $\mathcal{T}$ has a negative eigenvalue $E < 0$. Let ψ, ψ_1, c be as in the first step and $\xi := L^- \psi_1$. Then

$$\begin{aligned} 0 \quad & < \langle L^+ \xi, \xi \rangle = \langle L^+_\mu L^- \psi_1, L^- \psi_1 \rangle \\ & = \langle E(\psi_1 + c\phi_\mu), L^- \psi_1 \rangle \\ & = E \| (L^-)^{1/2} \psi_1 \|^2_{L^2} \qquad \text{because } L^- \phi_\mu = 0 \\ & < 0 : \text{contradiction.} \end{aligned}$$

Therefore, the eigenvalues of $\mathcal{T}$ are $\geqslant 0$.

Fourth step: Conclusion. Thanks to (77) and the second and third steps, the eigenvalues of $\mathcal{L}^2$ are non positive real numbers. Thus, the eigenvalues of $\mathcal{L}$ are purely imaginary. $\square$

B.4 Statement *(iv)*

Note that 0 is an eigenvalue of $\mathcal{M}_{\mp\pi^2}$ with multiplicity 2:

$$\mathcal{M}_{\mp\pi^2} \begin{pmatrix} \sin(\pi x) \\ 0 \end{pmatrix} = 0, \qquad \mathcal{M}_{\mp\pi^2} \begin{pmatrix} 0 \\ \sin(\pi x) \end{pmatrix} = 0$$

and the non zero eigenvalues of $\mathcal{M}_{-\pi^2}$ are $\{\pm(n^2-1)\pi^2; n \geqslant 2\}$:

$$\mathcal{M}_{\mp\pi^2} \begin{pmatrix} \sin(n\pi x) \\ 0 \end{pmatrix} = (n^2-1)\pi^2 \begin{pmatrix} \sin(n\pi x) \\ 0 \end{pmatrix},$$

$$\mathcal{M}_{\mp\pi^2} \begin{pmatrix} 0 \\ \sin(n\pi x) \end{pmatrix} = -(n^2-1)\pi^2 \begin{pmatrix} 0 \\ \sin(n\pi x) \end{pmatrix}.$$

For $\mu_0 \in [\mp\pi^2, +\infty)$, $\mathcal{M}_\mu$ converges to $\mathcal{M}_{\mu_0}$ when $\mu \to \mu_0$ in the sense of the generalized convergence of closed operators (i.e. convergence of the graph, see [32, Chapter IV, paragraph 2, page 197]). Thus $\mu \mapsto \beta_{n,\mu}$ is continuous for every $n \in \mathbb{N}$ (see [32, Chapter IV, paragraph 3.5]). $\square$

B.5 Statement *(v)*

In this section, we omit μ in subscript to simplify the notations. Let $n \in \mathbb{N}^*$. The map

$$\left| \begin{array}{ccc} \mathrm{Ker}(\mathcal{L} - i\beta_n \mathrm{Id}) & \to & \mathbb{C}^2 \\ \Phi & \mapsto & \Phi'(0) \end{array} \right.$$

is injective thanks to the uniqueness in Cauchy-Lipschitz theorem. Thus $\dim[\mathrm{Ker}(\mathcal{L} - i\beta_n \mathrm{Id})] \leqslant 2$.

Now, we prove by contradiction that no non zero eigenvalue possesses a generalized eigenvector. This proof follows the one of [42] for NLS on the whole space. We use the operator $\mathcal{T}$ introduced in (77). We assume that $\mathcal{L}_\mu$ has a generalized eigenvector associated to a non zero eigenvalue.

First step: $\mathcal{T}$ has a generalized eigenvector associated to a non zero eigenvalue. Let $\psi, \rho \in D(\mathcal{L}_\mu) - \{0\}$ and $E \neq 0$ be such that

$$\mathcal{L}_\mu \rho = E\rho, \qquad \mathcal{L}_\mu \psi = E\psi + \rho.$$

Then,

$$(\mathcal{L}_\mu^2 - E^2)\rho = 0, \qquad (\mathcal{L}_\mu^2 - E^2)\psi = 2E\psi,$$

thus $\mathcal{L}_\mu^2$ has a generalized eigenvector associated to the eigenvalue $2E$, and so has $\mathcal{T}$ (see 77).

Second step: $(L^-)^{1/2} L^+ (L^-)^{1/2}$ has a generalized eigenvector. Let $\psi, \rho \in D(\mathcal{T}) - \{0\}$, $E \neq 0$ be such that

$$\mathcal{T}\psi = E\psi, \qquad \mathcal{T}\rho = E\rho + c\psi.$$

Then ψ_1 and ρ_1 are not collinear to ϕ_μ (otherwise E would be zero). Let ψ_1, ρ_1 be the projections of ψ, ρ orthogonally to ϕ_μ. Then $(L^-)^{1/2}\psi_1 \neq 0$, $(L^-)^{1/2}\rho_1 \neq 0$ (because of (73)) and

$$[(L^-)^{1/2} L^+ (L^-)^{1/2} - E](L^-)^{1/2}\psi_1 = 0 \qquad [(L^-)^{1/2} L^+ (L^-)^{1/2} - E](L^-)^{1/2}\rho_1 = c(L^-)^{1/2}\psi_1.$$

thus $(L^-)^{1/2} L^+ (L^-)^{1/2}$ has a generalized eigenvector.

Third step: The operator $B := (L^-)^{1/2} L^+ (L^-)^{1/2}$ with domain $D(B) := H^4_{(0)}(0,1)$ is self adjoint, which gives the contradiction. The symmetry of B is obvious. Let us prove that $D(B^*) = D(B)$. Let $g, h \in L^2(0,1)$ be such that

$$\langle (L^-)^{1/2} L^+ (L^-)^{1/2} f, g \rangle = \langle f, h \rangle, \quad \forall f \in H^4_{(0)}(0,1).$$

Our goal is to prove that $g \in H^4_{(0)}(0,1)$. Taking $f \in \mathrm{Ker}[(L^-)^{1/2}]$ shows that $h \in \mathrm{Ker}[(L^-)^{1/2}]^\perp$. By Fredholm alternative applied to the self adjoint operator $(L^-)^{1/2}$, there exists $h_1 \in D[(L^-)^{1/2}] = H^1_0(0,1)$ such that $h = (L^-)^{1/2} h_1$. Then

$$\langle (L^-)^{1/2} L^+ (L^-)^{1/2} f, g \rangle = \langle f, (L^-)^{1/2}[h_1 + c\phi_\mu] \rangle, \quad \forall f \in H^4_{(0)}(0,1), \quad \forall c \in \mathbb{C}.$$

By self-adjointness of $(L^-)^{1/2}$ and (73), this gives

$$\langle (L^-)^{1/2} L^+ f_1, g \rangle = \langle f_1, h_1 + c\phi_\mu \rangle, \quad \forall f_1 \in H^3_{(0)}(0,1) \text{ with } f_1 \perp \phi_\mu, \quad \forall c \in \mathbb{C}.$$

The restriction '$f_1 \perp \phi_\mu$' may be removed by choosing

$$c := \frac{1}{\|\phi_\mu\|_{L^2}^2} \Big(\langle (L^-)^{1/2} L^+ \phi_\mu, g \rangle - \langle \phi_\mu, h_1 \rangle \Big).$$

Then,

$$\langle (L^-)^{1/2}L^+f_1, g\rangle = \langle f_1, h_1 + c\phi_\mu\rangle, \quad \forall f_1 \in H^3_{(0)}(0,1). \tag{81}$$

Thanks to (73), the operator $L^+ : H^3_{(0)}(0,1) \to H^1_0(0,1)$ is bijective and selfadjoint thus

$$\langle (L^-)^{1/2}f_2, g\rangle = \langle f_2, (L^+)^{-1}[h_1 + c\phi_\mu]\rangle, \quad \forall f_2 \in H^1_0(0,1).$$

By selfadjointness of $(L^-)^{1/2}$, this proves that

$$(L^-)^{1/2}g = (L^+)^{-1}[h_1 + c\phi_\mu]$$

belongs to $H^3_{(0)}(0,1)$ (because $h_1 \in H^1_0(0,1)$), thus $g \in H^4_{(0)}(0,1)$. $\square$

B.6 Statements *(vi)* and *(vii)*

One easily checks that (41) holds.

First step: Ker$(\mathcal{L}_\mu) = \mathbb{C}\Phi_0^+$, $\forall \mu \in (\mp\pi^2, +\infty)$. Let $(u,v) \in D(\mathcal{L}_\mu)$ be such that $L^-_\mu v = L^+_\mu u = 0$. From (73), we deduce that $u = 0$ and $v = c\phi_\mu$ for some $c \in \mathbb{R}$.

Second step: $\mathcal{L}_\mu$ does not have a third (linearly independent) generalized eigenvector, for every $\mu \in (\mp\pi^2, +\infty)$. We assume that there exists $(u,v) \in D(\mathcal{L}_\mu)$ such that $L^-_\mu v = \partial_\mu\phi_\mu$ and $L^+_\mu u = 0$. Then, thanks to (16) and the selfadjointness of L^-_μ, we get

$$0 < \langle \partial_\mu\phi_\mu, \phi_\mu\rangle = \langle L^-_\mu v, \phi_\mu\rangle = \langle v, L^-_\mu\phi_\mu\rangle = 0,$$

which is impossible.

This proves that (Φ_0^-, Φ_0^+) form a basis of the generalized null space for $\mathcal{L}_\mu$. The case of $\mathcal{L}^*_\mu$ may be treated similarly. Moreover, we have

$$\sigma i\beta_{m,\mu}\langle \Phi^\sigma_m, \Psi^\tau_n\rangle = \langle \mathcal{L}_\mu\Phi^\sigma_m, \psi^\tau_n\rangle = \langle \Phi^\sigma_m, \mathcal{L}^*_\mu\psi^\tau_n\rangle = \tau i\beta_{n,\mu}\langle \Phi^\sigma_m, \Psi^\tau_n\rangle.$$

This proves (44) when all the positive eigenvalues of $\mathcal{L}_\mu$ are simple.

C Analyticity of eigenvalues: proof

The proof of Proposition 26 relies on the fact that the dimension of the eigenspaces of $\mathcal{M}_\mu$ is at most two, and the following elementary result.

Proposition 29 *Let $I \subset \mathbb{R}$ be an interval and $B : I \to \mathcal{M}_2(\mathbb{R})$ be an analytic function. Assume that the eigenvalues of $B(\mu)$ are real for every $\mu \in I$. Then, there exists analytic functions $\lambda_1, \lambda_2 : I \to \mathbb{R}$ such that $Sp[B(\mu)] = \{\lambda_1(\mu), \lambda_2(\mu)\}$ for every $\mu \in I$.*

Proof of Proposition 29: The eigenvalues of $B(\mu)$ are

$$\lambda_\pm(\mu) := \frac{1}{2}\Big(\mathrm{Tr}[B(\mu)] \pm \sqrt{\Delta(\mu)}\Big) \text{ where } \Delta(\mu) := \mathrm{Tr}[B(\mu)]^2 - 4\mathrm{Det}[B(\mu)]. \tag{82}$$

Let $\mu_0 \in I$. If $\Delta(\mu_0) > 0$, then the previous formula defines 2 analytic functions on a neighborhood of μ_0. Let us assume that $\Delta(\mu_0) = 0$. Notice that $\Delta(\mu) \geqslant 0$, $\forall \mu \in I$ because $A(\mu)$ has real eigenvalues. Expanding $\Delta(\mu)$ in power series of $(\mu - \mu_0)$, we find $k \in \mathbb{N}^*$ and an function $\widetilde{\Delta}(\mu)$, analytic in a neighborhood of μ_0 and satisfying $\widetilde{\Delta}(\mu_0) > 0$ such that $\Delta(\mu) = (\mu - \mu_0)^{2k}\widetilde{\Delta}(\mu)$ on a neighborhood of μ_0. Then we get the conclusion with the formula

$$\lambda_1(\mu) := \frac{1}{2}\Big(\mathrm{Tr}[B(\mu)] - (\mu - \mu_0)^k\sqrt{\widetilde{\Delta}(\mu)}\Big), \quad \lambda_2(\mu) := \frac{1}{2}\Big(\mathrm{Tr}[B(\mu)] + (\mu - \mu_0)^k\sqrt{\widetilde{\Delta}(\mu)}\Big). \square$$

Proposition 30 *Let $\mu_0 \in (\mp\pi^2, \infty)$ and $n \in \mathbb{N}^*$. There exists an analytic function φ, defined on an open neighborhood I of μ_0 such that $\varphi(\mu_0) = \beta_{n,\mu_0}$ and $\varphi(\mu) \in Sp(\mathcal{M}_\mu)$, $\forall \mu \in I$.*

Proof of Proposition 30: Let $\mu_0 \in (\mp\pi^2, \infty)$.

First step: Reduction to a finite dimensional space.
This step follows exactly [32, Chap VII, Paragraph 1.3, proof of Theorem 1.7, page 368]. Let $\mathcal{C}$ be a closed curve in the complex plane that separates $\mathrm{Sp}(\mathcal{M}_{\mu_0})$ into two parts: a finite one $\Sigma'(\mu_0)$, with cardinal $N \in \mathbb{N}^*$ and an infinite one $\Sigma''(\mu_0)$. Since $\mathcal{M}_\mu$ converges to $\mathcal{M}_{\mu_0}$ when $\mu \to \mu_0$, in the generalized sense (convergence of graphs of closed operators), then, for sufficiently small $|\mu - \mu_0|$, $\mathrm{Sp}(\mathcal{M}_\mu)$ is likewise separated by $\mathcal{C}$ into a finite part $\Sigma'(\mu)$, with cardinal N, and an infinite part $\Sigma''(\mu)$, associated to the decomposition $L^2((0,1),\mathbb{C}^2) = E'(\mu) \oplus E''(\mu)$. The projection on $E'(\mu)$ along $E''(\mu)$ is given by

$$P(\mu) = \frac{1}{2\pi i}\int_{\mathcal{C}} (\mathcal{M}_\mu - z\mathrm{Id})^{-1} dz.$$

It is a bounded-holomorphic operator near $\mu = \mu_0$.

Let us construct a transformation $U(\mu)$ such that

(i) $U(\mu)$ and $U(\mu)^{-1}$ are bounded-holomorphic on $L^2((0,1),\mathbb{C}^2)$,

(ii) $U(\mu)P(\mu_0)U(\mu)^{-1} = P(\mu)$ for every μ near μ_0.

We define $U(\mu)$ and $V(\mu)$ as the operators on $L^2((0,1),\mathbb{C}^2)$, solutions of the linear ordinary differential equations

$$\left\{\begin{array}{l} U'(\mu) = Q(\mu)U(\mu), \\ U(\mu_0) = \mathrm{Id}, \end{array}\right. \qquad \left\{\begin{array}{l} V'(\mu) = -V(\mu)Q(\mu), \\ V(\mu_0) = \mathrm{Id}, \end{array}\right.$$

where $Q(\mu) := P'(\mu)P(\mu) - P(\mu)P'(\mu)$. Then, $U(\mu)$ and $V(\mu)$ are bounded-holomorphic and

$$(VU)' = V'U + VU' = -VQU + VQU = 0$$

thus $V(\mu)U(\mu) \equiv \mathrm{Id}$. This proves the announced properties on $U(\mu)$.

Note that

$$\hat{\mathcal{M}}_\mu := U(\mu)^{-1}\mathcal{M}_\mu U(\mu)$$

commutes with $P(\mu_0)$. Indeed, $\mathcal{M}_\mu$ commutes with $P(\mu)$ thus the property (ii) above proves

$$\begin{aligned} \hat{\mathcal{M}}_\mu P(\mu_0) &= U(\mu)^{-1}\mathcal{M}_\mu P(\mu)U(\mu) = U(\mu)^{-1}P(\mu)\mathcal{M}_\mu U(\mu) \\ &= P(\mu_0)U(\mu)^{-1}\mathcal{M}_\mu U(\mu) = P(\mu_0)\hat{\mathcal{M}}_\mu. \end{aligned}$$

Thus, the N-dimensional space $E'(\mu_0) = \mathrm{Range}[P(\mu_0)]$ is stable by $\hat{\mathcal{M}}_\mu$ and

$$\mathrm{Sp}[\hat{\mathcal{M}}_\mu|_{E'(\mu_0)}] = \Sigma'(\mu). \tag{83}$$

Second step: Analyticity of eigenvalues.
Let $n \in \mathbb{N}^*$. We apply the first step with a positively oriented circle $\mathcal{C}$ with center β_{n,μ_0} and radius $\epsilon > 0$ small enough so that $\mathcal{C}$ contains no other eigenvalue of $\mathcal{M}_{\mu_0}$. If β_{n,μ_0} is simple, then the previous construction shows that $\mu \mapsto \beta_{n,\mu}$ is analytic near $\mu = \mu_0$. Let us assume that β_{n,μ_0} is a multiple eigenvalue of $\mathcal{M}_{\mu_0}$. Thanks to Proposition 13 *(v)*,

$E'(\mu_0) := \mathrm{Ker}[\mathcal{M}_{\mu_0} - \beta_{n,\mu_0}\mathrm{Id}]$ has dimension 2. Let (e_1, e_2) be a basis of $E'(\mu_0)$. One may assume that e_1 and e_2 are real-valued functions, otherwise consider $(e_j + \overline{e_j})/2$ and $(e_j - \overline{e_j})/(2i)$. Let $B(\mu)$ be the $2 * 2$-matrix of the operator $\hat{\mathcal{M}}_\mu|_{E'(\mu_0)}$ in the basis (e_1, e_2). Then $B(\mu)$ is analytic and has only real valued eigenvalues, thanks to (83) and Proposition 13 *(ii)*. Let us prove that $B(\mu)$ has real valued coefficients, which allows to conclude thanks to Proposition 29.

Step 2.1: We prove that $P(\mu)$ is real valued, i.e. $P(\mu)f \in L^2((0,1),\mathbb{R}^2)$, $\forall f \in L^2((0,1),\mathbb{R}^2)$. Indeed,

$$\begin{aligned} P(\mu)f &= \frac{1}{2\pi i}\int_{\mathcal{C}}(\mathcal{M}_\mu - z\mathrm{Id})^{-1} f dz = \frac{1}{2\pi}\int_0^{2\pi}\left(\mathcal{M}_\mu - (\beta_{n,\mu_0} + \epsilon e^{i\theta})\mathrm{Id}\right)^{-1} f \epsilon e^{i\theta} d\theta \\ &= \overline{\frac{1}{2\pi}\int_0^{2\pi}\left(\mathcal{M}_\mu - (\beta_{n,\mu_0} + \epsilon e^{-i\theta})\mathrm{Id}\right)^{-1} f \epsilon e^{-i\theta} d\theta} \\ &= \overline{\frac{1}{2\pi i}\int_{\mathcal{C}}(\mathcal{M}_\mu - z\mathrm{Id})^{-1} f dz} = \overline{P(\mu)f}. \end{aligned}$$

Step 2.2: We prove that $U(\mu)$ and $U(\mu)^{-1}$ are real valued. Indeed, if $f \in L^2((0,1),\mathbb{R}^2)$, then $g(\mu) := U(\mu)f$ and is the solution of the ordinary differential equation

$$\begin{cases} g'(\mu) = Q(\mu)g(\mu), \\ g(\mu_0) = f, \end{cases}$$

thus it is real valued, thanks to Step 2.1 and the uniqueness in Cauchy-Lipschitz theorem.

Step 2.3: We prove that $B(\mu)$ have real valued coefficients. Thanks to *Step 2.2*, we have

$$B(\mu)e_j = \hat{\mathcal{M}}_\mu e_j = U(\mu)^{-1}\mathcal{M}_\mu U(\mu)e_j \in L^2((0,1),\mathbb{R}^2), \quad \forall j = 1,2.$$

thus its coefficients on the (real-valued) basis (e_1, e_2) are real. □

Proof of Proposition 26: By [32, Chapter VII, paragraph 3, Theorem 1.8], the eigenvalues of $\mathcal{M}_\mu$ are branches of one or several analytic functions, which have only algebraic singularities, and which are everywhere continuous. An exceptional point μ_0 is

- either a branch point (see [32, Chap II, Paragraph 1.2] for a definition),
- or a regular point where different eigenvalues coincide (crossing).

Moreover, when we consider a finite number of eigenvalues, there are only a finite number of exceptional points μ_0 in each compact set of $(\mp\pi^2, \infty)$. Proposition 30 shows that there are no branch point and that eigenvalues can be followed analytically through crossings.

Let $n \in \mathbb{N}^*$. There exists $\delta > \mp\pi^2$ such the map $\mu \mapsto \beta_{n,\mu}$ is continuous on $[\mp\pi^2, \delta)$, and $\beta_{n,\mu}$ is a simple eigenvalue of $\mathcal{M}_\mu$ for every $[\mp\pi^2, \delta)$. Then, $\mu \mapsto \beta_{n,\mu}$ is analytic on $(\mp\pi^2, \delta)$ thanks to Proposition 30. Let μ^* be the sup of the $\mu_\sharp \geqslant \delta$ such that $\mu \in (\mp\pi^2, \delta) \mapsto \beta_{n,\mu}$ may be extended in an analytic function $\varphi : (\mp\pi^2, \mu_\sharp) \to \mathbb{R}$, which is everywhere an eigenvalue of $\mathcal{M}_\mu$. We prove by contradiction that $\mu^* = \infty$.

We assume that $\mu^* < +\infty$. Then at most a finite number of crossings may happen on $(\mp\pi^2, \mu^*)$: there exists a finite number $N \in \mathbb{N}$ of points $\mu_1, ..., \mu_N \in (\delta, \mu^*)$ such that $\varphi(\mu)$ coincide with different eigenvalues $\beta_{n_{k-1},\mu}$ when $\mu < \mu_k$ and $\beta_{n_k,\mu}$ when $\mu > \mu_k$, with $n_k = n_{k-1} \pm 1$, for $k = 1, ..., N$. In particular, for $\mu \in (\mu_N, \mu^*)$, we have $\varphi(\mu) = \beta_{n_N,\mu}$. Thanks to Proposition 30, $\varphi(\mu)$ may be extended into an analytic function on a larger interval than $(\mp\pi^2, \mu^*)$, that is everywhere an eigenvalue of $\mathcal{M}_\mu$, which is impossible. Therefore $\mu^* = \infty$ and Proposition 26 is proved. □

D Moment problem

The following proposition is crucial in the controllability of the linearized system. It is a consequence of the Ingham inequality proved by Haraux in [28] and may be proved exactly as [12, Corollary 2 in Appendix B].

Proposition 31 *Let $T>0$, $N\in\mathbb{N}$ and $(\omega_k)_{k\in\mathbb{N}}$ be an increasing sequence of $(0,+\infty)$ such that $\omega_{k+1}-\omega_k\to+\infty$ when $k\to+\infty$. Then there exists a continuous linear map*

$$\left| \begin{array}{cccc} L: & \mathbb{R}^N\times l^2(\mathbb{N}^*,\mathbb{C}) & \to & L^2((0,T),\mathbb{R}) \\ & (\tilde{d},d) & \mapsto & L(\tilde{d},d) \end{array} \right.$$

such that, for every $\tilde{d}=(\tilde{d}_1,...,\tilde{d}_N)\in\mathbb{R}^N$ and $d=(d_k)_{k\in\mathbb{N}}\in l^2(\mathbb{N}^*,\mathbb{C})$, the function $v:=L(\tilde{d},d)$ solves

$$\begin{array}{l} \int_0^T v(t)e^{i\omega_k t}dt = d_k, \forall k\in\mathbb{N}^*, \\ \int_0^T t^k v(t)dt = \tilde{d}_{k+1}, \forall k=0,...,N-1. \end{array}$$

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